Non-monotonic Reasoning via Dynamic Consequence

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Abstract. We approach the concept of Pivotal Rule Consequence (PRC) proposed in [14,15] from a semantical perspective, resorting to model updates in Public Announcement Logic (PAL) [17]. In doing this, we take inspiration from the notion of dynamic consequence from [3,6]. Our perspective gains in interest since PRC serves as a "bridge" from Classical Logic to Default Logic –one of the most well-known non-monotonic formalisms. We show how the internalization of PRC in PAL leads to clear semantics of the former, and to completeness and transfer results. Moreover, we address the case of credulous consequence in Default Logic as a particular case of PRC. Interestingly, we cast credulous consequence as a model checking problem. We argue that our results open the way to use well-known semantic tools from modal logic to study properties of different non-monotonic logics.

1 Introduction

In spite of its logical terminology, the field of non-monotonic logic has a reputation for presenting itself in an unfamiliar way to traditional logicians. In [14,15], it is shown that this need not be the case. The argument put forth is that, provided some preconceptions and misconceptions are set aside, there are logics acting as natural "bridges" between classical consequence and some well-known non-monotonic logics. These bridges possess some of the characteristics of their classical starting points and, at the same time, they start exhibiting some of the distinctive features of their non-monotonic endpoints. Crucially, monotonicity is not given up initially. Instead, it originates as a result of introducing "consistency checks". Using these bridges, it is possible to transition in a clear and step-wise manner from classical to non-monotonic logics.

In this paper, we focus our attention on so-called *pivotal-rule consequence* (PRC). This bridge extends the classical notion of consequence by using non-admissible rules of inference called *pivotal rules* (i.e., by using rules whose incorporation modifies the set of consequences of a logic). By imposing "consistency checks" on how pivotal rules are used, PRC takes us from classical consequence to consequence in Default Logic [18] –one of the most prominent approaches to non-monotonic reasoning [1].

The results in [14,15] are obtained by manipulating PRC via closure operators on sets of formulas. In this work, we revise PRC from a semantical perspective resorting to the formal machinery of Public Announcement Logic (PAL) [17]. In this revision, we take inspiration from van Benthem's program on "logical dynamics" [3] (see also [6,19]). On the one hand, PAL is a modal logic with an *announcement* modality $[\varphi]$ that restricts the current (relational) model to those states where φ holds. On the other hand, the program on logical dynamics shifts the static role played by premisses in a consequence relation and views them as new information brought into consideration via some action. This results in the consequence relation taking a dynamic stance. In very simple terms, the program on logical dymamics takes the premisses of a consequence relation as a sequence of announcements; i.e., $\psi_1, \ldots, \psi_n \vDash \varphi$ shifts to $\vDash [\psi_1] \ldots [\psi_n]\varphi$. We adapt these ideas to capture the effect brought in by the use of pivotal rules in PRC. In doing so, we show that pivotal-rule consequence can be seen as a dynamic modal logic.

In addressing PRC from a semantic perspective we complete the picture of [14,15]. Our results establish a novel connection between PRC and Modal Logic, via PAL. In addition to offering new insights into the nature of the framework proposed n [14,15], our approach allows for transfer results: theoretical and practical tools developed for PAL turn out relevant for PRC (and for Default Logic as a particular case). These results include: bisimulations, axiomatizations, tableaux calculi, model checking algorithms, etc.

Contributions. We show how PRC, and credulous consequence in Default Logic as a particular case, can be seen as a modal logic using dynamic update operators. In doing this, the formal machinery of PAL turns out to be sufficiently expressive. Interestingly, we obtain that credulous consequence reduces to *model checking* in PAL. Our results establish that PAL can be seen as providing a model theoretic semantics for PRC and to credulous consequence as a special case. This approach differs fundamentally from other semantic takes on Default Logic, e.g., those in [9,4,7]. We provide a mapping from credulous consequence to consequence in a modal logic with updates, i.e., PAL. In particular, we capture credulous consequence as particular tautologies of PAL–something which, as far as we know, has not been addressed. Moreover, we detail how our approach leads to natural axiomatizations, and to expressivity and complexity results. Concretely, our contributions consist of:

1. A novel characterization of PRC in terms of Hilbert-style deductions (something which we use in the rest of the paper).
2. A characterization of PRC as particular tautologies of an alternative presentation of PAL, which we refer to as Pivotal Modal Logic (PML).
3. A study of the logical properties of PRC via transfer results from PML. In particular, axiomatization, expressivity and complexity results.
4. A characterization of credulous consequence in PML. Again, we use our framework to investigate an axiomatization, expressivity and complexity results. Interestingly, deciding credulous consequence boils down to a model checking problem over a particular model. This enables us to obtain a reasonable small complexity upper bound in a simple way.

Structure. Sec. 2 covers basic material on Classical Propositional Logic. Sec. 3 presents PRC. As a first novel result, we offer a way of looking at PRC via Hilbert-style deductions. Sec. 4 shows how PAL can be used to capture PRC from a semantic perspective (see Thm. 2). We use our framework to study: axiomatization, expressivity and complexity results. Sec. 4.2 discusses the case of credulous consequence from a semantic perspective (see Thm. 5) and study its properties. Sec. 5 offers some final remarks and discusses future lines of research.

2 Preliminaries

We briefly review the basics of classical propositional logic (CPL; see, e.g., [16]).

Definition 1. *Let* $\mathsf{Prop} = \{\, p_i \mid i \in \mathbb{N} \,\}$ *be a set of* proposition symbols*; the set* Form_0 formulas *of* CPL *built on* Prop *is given by the grammar*

$$\varphi, \psi ::= p_i \mid \bot \mid \varphi \to \psi.$$

We use $\top$, $\neg\varphi$, $\varphi \vee \psi$, $\varphi \wedge \psi$, *and* $\varphi \leftrightarrow \psi$ *as abbreviations – defined as usual.*

The semantics of CPL is given resorting to assignments and valuations.

Definition 2. *An* assignment *is a function* $v : \mathsf{Prop} \to \{0, 1\}$*. A* valuation *is an extension of an assignment* v *to a function* $v^* : \mathsf{Form}_0 \to \{0, 1\}$ *s.t.:* $v^*(\bot) = 0$, *and* $v^*(\varphi \to \psi) = 1$ *iff either* $v^*(\varphi) = 0$, *or* $v^*(\psi) = 1$*. A valuation* v^* *is a* model *of* φ *(or* φ *is* satisfiable *in* v^**), written* $v^* \vDash \varphi$, *iff* $v^*(\varphi) = 1$*. A formula* φ *has a model, or is satisfiable, iff exists a valuation* v^* *s.t.* $v^* \vDash \varphi$*. These definitions extend to sets of formulas in the obvious way.*

Def. 2 yields a relation of *semantic consequence* between sets of formulas, called premisses, and formulas, called consequences. This relation is given below.

Definition 3. *A formula* φ *is a* (semantic) consequence *of a set of formulas* Φ, *written* $\Phi \vDash \varphi$, *iff for all valuations* v^*, $v^* \vDash \Phi$ *implies* $v^* \vDash \varphi$*. We call a formula* φ *a* tautology*, and write* $\vDash \varphi$, *iff* $\emptyset \vDash \varphi$*.*

The syntactic counterpart of the relation $\vDash$ is defined via deductions.

Definition 4. *A* deduction of a formula φ from a set of formulas Φ *is a finite sequence* $\bar{\psi} = \psi_1 \dots \psi_n$ *of formulas s.t.* $\psi_n = \varphi$, *and for all* $k < n$, *either:*

1. ψ_k *is an axiom of* CPL *(see [16]);*
2. ψ_k *is a premiss (i.e.,* $\psi_k \in \Phi$*);*
3. ψ_k *is obtained using* mp *(i.e., exists* $i, j < k$ *s.t.* $\psi_j = \psi_i \to \psi_k$*).*

We write $\Phi \vdash \varphi$ *iff exists a deduction of* φ *from* Φ*.*

Theorem 1 (Soundness-Completeness). $\Phi \vdash \varphi$ *iff* $\Phi \vDash \varphi$*.*

3 From Classical to Default Consequence

Deductions in CPL give rise to an operation $\mathrm{Cn} : 2^{\mathsf{Form}_0} \to 2^{\mathsf{Form}_0}$ defined as $\mathrm{Cn}(\Phi) = \{ \varphi \mid \Phi \vdash \varphi \}$. This operation satisfies some well-known properties. In particular, it satisfies the *principle of monotonicity*, i.e.:

$$\Phi \subseteq \Psi \text{ implies } \mathrm{Cn}(\Phi) \subseteq \mathrm{Cn}(\Psi).$$

Following [14,15], we show how to construct, in a natural way, an operation D which extends Cn but which fails to satisfy monotonicity. We introduce D by taking a detour via an operation C which retains monotonicity, but starts exhibiting some of the distinctive features of D. When seen in this light, C acts as a bridge between Cn and D, and it streamlines the presentation of the latter.

We begin the presentation of the operation C by introducing *pivotal rules*.

Definition 5. *A* pivotal rule *is a pair* (π, χ) *of formulas of* CPL. *We use* (π/χ) *as notation for a pivotal rule; we call* π *its* prerequisite *and* χ *its* consequent. *The* image *of a set* P *of pivotal rules under a set* Φ *formulas of* CPL *is defined as* $P(\Phi) = \{ \chi \mid (\pi/\chi) \in P \text{ and } \pi \in \Phi \}$. *The set* Φ is closed under P *if* $P(\Phi) \subseteq \Phi$.

Henceforth, we will deal only with finite sets P of pivotal rules. Pivotal rules are best understood as non-admissible rules of inference, i.e., rules whose incorporation extends the set of CPL-consequences of a set of formulas. Def. 6 makes this idea clear.

Definition 6. *Let* P *be a set of pivotal rules; define* $\mathrm{C}_P : 2^{\mathsf{Form}_0} \to 2^{\mathsf{Form}_0}$ *as:*

$$\mathrm{C}_P(\Phi) = \bigcap \{ \Psi \mid \Phi \subseteq \Psi,\ \mathrm{Cn}(\Psi) \subseteq \Psi,\ \text{and } P(\Psi) \subseteq \Psi \}.$$

We say that φ *is a* pivotal-rule consequence *of* Φ *iff* $\varphi \in \mathrm{C}_P(\Phi)$.

The operation C_P in Def. 6 is called *pivotal-rule consequence* in [15]. From now on, we refer to it as *pivotal consequence*. In words, $\mathrm{C}_P(\Phi)$ is the smallest superset of Φ closed under Cn and P. Note that Def. 6 does not introduce a single operation C but a family of such operations; one for each set P of pivotal rules. We drop the subscript P if it can be understood from context or not needed.

From its definition, $\mathrm{Cn}(\Phi) \subseteq \mathrm{C}(\Phi)$. For this reason, we say that C extends Cn. Just as Cn, C is monotonic: $\Phi \subseteq \Psi$ implies $\mathrm{C}(\Phi) \subseteq \mathrm{C}(\Psi)$. Moreover, C is closed: $\mathrm{C}(\mathrm{C}(\Phi)) = \mathrm{C}(\Phi)$. Nevertheless, C diverges from Cn in some important ways. For instance, C may fail to satisfy the property of *disjunction*. This property states that if $\gamma \in \mathrm{C}(\{\varphi\})$ and $\gamma \in \mathrm{C}(\{\psi\})$, then, $\gamma \in \mathrm{C}(\{\varphi \vee \psi\})$. To see its failure, let $P = \{(p/q), (r/q)\}$; then $q \in \mathrm{C}_P(\{p\})$ and $q \in \mathrm{C}_P(\{r\})$, but $q \notin \mathrm{C}_P(\{p \vee r\})$.

Def. 7 offers another way of looking at C. This definition also serves as an intermediate step towards non-monotonicity.

Definition 7. *Let* $\bar{P}$ *be a total ordering of a set* P *of pivotal rules; we use* (π_i/χ_i) *to indicate the* i*-th element of* $\bar{P}$. *Define an operation* $\mathrm{C}_{\bar{P}} : 2^{\mathsf{Form}_0} \to 2^{\mathsf{Form}_0}$ *s.t. for all* Φ, $\mathrm{C}_{\bar{P}}(\Phi) = \bigcup \{ \mathrm{C}^i_{\bar{P}}(\Phi) \mid i \geq 0 \}$ *where:*

$$\mathrm{C}^0_{\bar{P}}(\Phi) = \mathrm{Cn}(\Phi) \quad \text{and} \quad \mathrm{C}^{(i+1)}_{\bar{P}}(\Phi) = \begin{cases} \mathrm{Cn}(\mathrm{C}^i_{\bar{P}}(\Phi) \cup \{\chi_i\}) & \text{if } \pi_i \in \mathrm{C}^i_{\bar{P}}(\Phi) \\ \mathrm{C}^i_{\bar{P}}(\Phi) & \text{otherwise.} \end{cases}$$

As before, we write $\bar{\mathrm{C}}$ when there is no need to make the set P of pivotal rules explicit. The operation $\bar{\mathrm{C}}$ does not immediately characterize the operation C. For instance, in general, $\bar{\mathrm{C}}$ is not closed, i.e., $\bar{\mathrm{C}}(\Phi) \neq \bar{\mathrm{C}}(\bar{\mathrm{C}}(\Phi))$. To see why, let $P = \{(q/r), (p/q)\}$; consider the total ordering $\bar{P}$ in which (q/r) is before (p/q); then, $\mathrm{C}_{\bar{P}}(\{p\}) = \mathrm{Cn}(\{p, q\})$; however, $\mathrm{C}_{\bar{P}}(\mathrm{C}_{\bar{P}}(\{p\})) = \mathrm{Cn}(\{p, q, r\})$. Nonetheless, it is possible to establish the following result.

Proposition 1. *$\varphi \in \mathrm{C}_P(\Phi)$ iff exists $\bar{P}$ s.t. $\varphi \in \mathrm{C}_{\bar{P}}(\Phi)$.*

Prop. 1 tells us that pivotal consequence amounts to checking what is the case in *some* total ordering of pivotal rules.

The argument in [15] is that C helps us to pave the way to non-monotonicity. More precisely, we obtain a non-monotonic consequence operation by imposing a consistency constraint on the use of the pivotal rules. We make this clear below.

Definition 8. *Let $\bar{P}$ be a total ordering on a set P of pivotal rules; define an operation $\mathrm{D}_{\bar{P}} : 2^{\mathsf{Form}_0} \to 2^{\mathsf{Form}_0}$ s.t. for all Φ, $\mathrm{D}_{\bar{P}}(\Phi) = \bigcup\{\, \mathrm{D}^i_{\bar{P}}(\Phi) \mid i \geq 0 \,\}$ where:*

$$\mathrm{D}^0_{\bar{P}}(\Phi) = \mathrm{Cn}(\Phi) \quad \textit{and} \quad \mathrm{D}^{(i+1)}_{\bar{P}}(\Phi) = \begin{cases} \mathrm{Cn}(\mathrm{D}^i_{\bar{P}}(\Phi) \cup \{\chi_i\}) & \textit{if } \pi_i \in \mathrm{D}^i_{\bar{P}}(\Phi) \\ & \quad \textit{and } \neg\chi_i \notin \mathrm{D}^i_{\bar{P}}(\Phi) \\ \mathrm{D}^i_{\bar{P}}(\Phi) & \textit{otherwise.} \end{cases}$$

Again, we use $\bar{\mathrm{D}}$ when there is no need to make P explicit. The operation $\bar{\mathrm{D}}$, and the operations Cn and $\bar{\mathrm{C}}$ are related in the following proposition.

Proposition 2. $\mathrm{Cn}(\Phi) \subseteq \bar{\mathrm{D}}(\Phi) \subseteq \bar{\mathrm{C}}(\Phi)$.

$\bar{\mathrm{D}}$ retains some properties of $\bar{\mathrm{C}}$. E.g., it extends Cn; in general it is not closed; and it may fail to satisfy the property of disjunction. This said, orderings which are indistinguishable in the case of $\bar{\mathrm{C}}$ are distinguishable in the case of $\bar{\mathrm{D}}$. To illustrate this point, let $P = \{(p/q), (p/\neg q)\}$; consider orderings $\bar{P}_1 = \langle (p/q), (p/\neg q)\rangle$ and $\bar{P}_2 = \langle (p/\neg q), (p/q)\rangle$; then $\mathrm{D}_{\bar{P}_1}(\{p\}) = \mathrm{Cn}(\{p, q\})$ and $\mathrm{D}_{\bar{P}_2}(\{p\}) = \mathrm{Cn}(\{p, \neg q\})$. Yet, $\mathrm{C}_{\bar{P}_1}(\{p\}) = \mathrm{C}_{\bar{P}_2}(\{p\}) = \mathsf{Form}_0$. More interestingly, $\bar{\mathrm{D}}$ fails to satisfy monotonicity, i.e., it is *non-monotonic*. For example, let $P = \{(p/q)\}$; then $\mathrm{D}_{\bar{P}}(\{p\}) = \mathrm{Cn}(\{p, q\})$; however, $\mathrm{D}_{\bar{P}}(\{p, \neg q\}) = \mathrm{Cn}(\{p, \neg q\})$.

The operation $\bar{\mathrm{D}}$ is in itself of interest. However, in some cases, we may want to depart from particular orderings. We make this idea precise in Def. 9.

Definition 9. *Define D_P s.t. $\varphi \in \mathrm{D}_P(\Phi)$ iff exists $\bar{P}$ s.t. $\varphi \in \mathrm{D}_{\bar{P}}(\Phi)$.*

We often write D instead of D_P. The following proposition is immediate.

Proposition 3. *It follows that: $\bar{D}(\Phi) \subseteq \mathrm{D}(\Phi) \subseteq \mathrm{C}(\Phi)$.*

The operation D is interesting because it coincides with *credulous* consequence for the case of normal defaults in default logic (see [18,14]). Namely, φ *is a credulous default consequence of* Φ iff $\varphi \in \mathrm{D}(\Phi)$. It is important to notice that this operation is not closed, i.e., in general, $\mathrm{D}(\mathrm{D}(\Phi)) \neq \mathrm{D}(\Phi)$. To see why, let $P = \{(\top/p), (\top/\neg p)\}$; it follows that $\mathrm{D}_P(\emptyset) = \mathrm{Cn}(\{p\}) \cup \mathrm{Cn}(\{\neg p\})$; whereas $\mathrm{D}_P(\mathrm{D}_P(\emptyset)) = \mathsf{Form}_0$. This example also shows that $\mathrm{D}(\emptyset) \neq \mathrm{Cn}(\mathrm{D}(\emptyset))$. In other words, D may not be closed under Cn.

Pivotal Deductions. We conclude this section with a novel characterization of the operations C and D via what we call pivotal deductions. This is our first contribution and it provides us the right setting to investigate pivotal consequence from a dynamic logic perspective. We will retake these ideas in Sec. 4.

Definition 10. *Let P be a set of pivotal rules; a P-*deduction *of a formula φ* from a set of formulas Φ *is a finite sequence $\bar{\psi} = \psi_1 \dots \psi_n$ of formulas such that $\psi_n = \varphi$, and for all $k < n$ one of the following holds:*

1. *ψ_k is an axiom of* CPL*, a premiss, or obtained using* mp *(see Def. 4); or*
2. *exists $j < k$ s.t. $(\psi_j/\psi_k) \in P$ –called P-*detachment*.*

We say that a pivotal rule $\rho \in P$ has been used in $\bar{\psi}$ iff $\rho = (\psi_j/\psi_k)$ for some $1 \leq j < k \leq n$. Let Ψ be the set of all consequents of the pivotal rules used in $\bar{\psi}$, we call $\bar{\psi}$ credulous *iff $\mathrm{Cn}(\Phi \cup \Psi) = \mathsf{Form}_0$ iff $\mathrm{Cn}(\Phi) = \mathsf{Form}_0$. We write:*

1. *$\Phi \vdash_P \varphi$ iff exists a P-deduction of φ from Φ;*
2. *$\Phi \mathrel{|\!\!\sim}_P \varphi$ iff exists a credulous P-deduction of φ from Φ.*

We often refer to P-deductions as *pivotal deductions* and to credulous P-deductions as *credulous deductions*. Intuitively, a pivotal deduction extends the notion of a deduction in CPL acommodating for the use of pivotal rules via P-detachment. The following propositions are immediate from the definitions.

Proposition 4. *$\Phi \vdash_P \varphi$ iff exists a seq. $\bar{\rho} = \rho_1 \dots \rho_n$ of pivotal rules in P s.t.:*

$$\Phi \cup X \vdash \varphi \text{ and for all } 1 \leq i \leq n,\ \Phi \cup X_{(i-1)} \vdash \pi_i, \tag{1}$$

where: $\pi_i = (\pi_i/\chi_i)$; $X_j = \{\chi_1, \dots, \chi_j\}$; and $X = X_n$. Moreover, $\Phi \mathrel{|\!\!\sim}_P \varphi$ iff in addition to Eq. (1) it holds that

$$\text{for all } 1 \leq i \leq n,\ \Phi \cup X_{(i-1)} \nvdash \neg\chi_i. \tag{2}$$

Proposition 5. *$\varphi \in \mathrm{C}_P(\Phi)$ iff $\Phi \vdash_P \varphi$; and $\varphi \in \mathrm{D}_P(\Phi)$ iff $\Phi \mathrel{|\!\!\sim}_P \varphi$;*

From Props. 4 and 5, it follows that pivotal deductions provide witnesses for $\varphi \in \mathrm{C}(\Phi)$; whereas credulous deductions provide witnesses for $\varphi \in \mathrm{D}(\Phi)$.

Remark 1. Def. 10 and Prop. 5 bring about a discussion on *compactness* for the operations C and D. It can easily be established that $\varphi \in \mathrm{C}(\Phi)$ iff for some finite $\Phi' \subseteq \Phi$, $\varphi \in \mathrm{C}(\Phi')$. The same claim is not true for D. Although $\varphi \in \mathrm{D}(\Phi)$ implies that there is a finite $\Phi' \subseteq \Phi$ s.t. $\varphi \in \mathrm{D}(\Phi')$, in general, the converse of this statement does not hold. This is unsurprising for, as commented on in [14], full compactness for D brings back monotonicity.

4 From Pivotal Rules to Model Updates

In this section we look at pivotal consequence through the lens of modal logic [5]. The modal logic we use is an alternative presentation of the single-agent Public Announcement Logic (PAL) [17]. PAL has as its distinguishing characteristic a modality which can "update" the model while a formula is evaluated. The inclusion of this update modality has three main benefits in our semantic exploration of pivotal consequence. First, it enables us to internalize the effect of pivotal rules in the object language, inspired by the notion of dynamic consequence [3,6]. Second, it enables us to provide a semantics for pivotal rules. Third, it opens the door for a study of operations C and D in Sec. 3 from the perspective of modal model theory. Our framework builds a novel bridge between pivotal consequence and modal logic, and particularly, from default consequence to modal logic.

4.1 Pivotal Consequence in PAL

We begin with a brief introduction to the formal machinery behind PAL [8]. To avoid confusions with its usual presentation, we refer to our presentation of PAL as *Pivotal Modal Logic* (PML). In what follows, we use $\vDash_{\mathsf{CPL}}$ to indicate the relation of semantic consequence of CPL (see Def. 3).

Definition 11. *The set* Form *of* PML *formulas is built over the set* Prop *of proposition symbols and is given by the grammar:*

$$\varphi, \psi ::= p_i \mid \bot \mid \varphi \to \psi \mid \mathsf{A}\varphi \mid [\varphi]\psi.$$

Other Boolean operators are defined as usual. Moreover, we use $\mathsf{E}\varphi$ *and* $\langle\varphi\rangle\psi$ *to abbreviate* $\neg\mathsf{A}\neg\varphi$ *and* $\neg[\varphi]\neg\psi$*, resp. We refer to the elements of the language of* PML *as* pivotal (modal) formulas*.*

Pivotal formulas are interpreted over pivotal models.

Definition 12. *A* pivotal model *is a non-empty subset* $\mathfrak{M} \subseteq \{0,1\}^{\mathsf{Prop}}$*, i.e., it is a non-empty set of functions* $w : \mathsf{Prop} \to \{0,1\}$*. The pair* $\mathfrak{M}, w$ *is a* pointed pivotal model *iff* $w \in \mathfrak{M}$*. We say that* $\mathfrak{M}, w$ *is* finite *iff* $\mathfrak{M}$ *is finite.*

Intuitively, we look at pivotal models as sets of CPL-assignments (cf. Def. 2). The definition of satisfiability in a pivotal model is given immediately below.

Definition 13. *Let* $\mathfrak{M}, w$ *be a pointed pivotal model; for any pivotal formula* φ*, we define* φ is satisfiable in $\mathfrak{M}$ at w*, written* $\mathfrak{M}, w \Vdash \varphi$*, inductively as:*

$$\begin{array}{lll} \mathfrak{M}, w \Vdash \bot & & \textit{never} \\ \mathfrak{M}, w \Vdash p & \textit{iff} & w(p) = 1 \\ \mathfrak{M}, w \Vdash \varphi \to \psi & \textit{iff} & \mathfrak{M}, w \Vdash \varphi \textit{ implies } \mathfrak{M}, w \Vdash \psi \\ \mathfrak{M}, w \Vdash \mathsf{A}\varphi & \textit{iff} & \textit{for all } w' \in \mathfrak{M},\ \mathfrak{M}, w' \Vdash \varphi \\ \mathfrak{M}, w \Vdash [\varphi]\psi & \textit{iff} & \mathfrak{M}, w \Vdash \varphi \textit{ implies } \mathfrak{M}|_\varphi, w \Vdash \psi, \end{array}$$

where: $\mathfrak{M}|_\varphi = \{\, w \in \mathfrak{M} \mid \mathfrak{M}, w \Vdash \varphi \,\}$[5]*. We write* $\mathfrak{M} \Vdash \varphi$ *iff for all* w *in* $\mathfrak{M}$*,* $\mathfrak{M}, w \Vdash \varphi$*. Let* Φ *be a set of pivotal formulas; we define* $\mathfrak{M}, w \Vdash \Phi$ *and* $\mathfrak{M} \Vdash \Phi$ *as*

[5] $\mathfrak{M}|_\varphi$ might be empty. If so, $\mathfrak{M}, w \Vdash \varphi$ doesn't hold and $\mathfrak{M}, w \models [\varphi]\psi$ is trivially true.

usual. We write $\Phi \vDash \varphi$ iff for all $\mathfrak{M}, w$, if $\mathfrak{M}, w \Vdash \Phi$, then, $\mathfrak{M}, w \Vdash \varphi$. We write $\vDash \varphi$ instead of $\emptyset \vDash \varphi$.

Def. 13 states that A and E are the universal and existential modalities [11]; and that $[\_]$ and $\langle\_\rangle$ are the update modalities of PAL [17].

Let us turn our attention onto how to use PML to capture pivotal consequence from a semantic perspective. The following propositions are preliminary and are meant to shed light into the relation between pivotal models and CPL valuations.

Proposition 6. *Let φ and ψ be CPL-formulas; then: 1. $\{\varphi\} \vDash_{\mathsf{CPL}} \psi$ iff $\{\varphi\} \vDash \psi$; and 2. $\{\varphi\} \vDash_{\mathsf{CPL}} \psi$ iff $\vDash [\varphi]\psi$. Moreover, for any PML-formula χ, we get that: 3. $\vDash [\varphi][\psi]\chi$ iff $\vDash [\varphi \wedge \psi]\chi$.*

Proof. Item 1 is immediate. For Item 2, we prove $\{\varphi\} \nvDash_{\mathsf{CPL}} \psi$ iff $\nvDash [\varphi]\psi$. Left to right: Suppose that $\{\varphi\} \nvDash_{\mathsf{CPL}} \psi$. Then, there is an assignment $v : \mathsf{Prop} \to \{0, 1\}$ s.t. $v^*(\varphi) = 1$ and $v^*(\psi) = 0$. Let $\mathfrak{M}$ be the pivotal model which consists solely of v, i.e., $\mathfrak{M} = \{v\}$. Since $\mathfrak{M}|_\varphi = \mathfrak{M} = \{v\}$, it follows that $\mathfrak{M}, v \Vdash \varphi$ and $\mathfrak{M}|_\varphi, v \nVdash \psi$. Thus, $\nvDash [\varphi]\psi$. Right to left: Suppose that $\nvDash [\varphi]\psi$. There is $\mathfrak{M}, w$ s.t. $\mathfrak{M}, w \nVdash [\varphi]\psi$. This means that $\mathfrak{M}, w \Vdash \varphi$ and $\mathfrak{M}|_\varphi, w \nVdash \psi$. Let w^* be the CPL-valuation based on w; immediately, $w^*(\varphi) = 1$ and $w^*(\psi) = 0$. Therefore, $\{\varphi\} \nvDash_{\mathsf{CPL}} \psi$. Item 3 follows from the fact that: if φ and ψ are CPL-formulas, then, for any pivotal model $\mathfrak{M}$, $(\mathfrak{M}|_\varphi)|_\psi = \mathfrak{M}|_{(\varphi\wedge\psi)}$. To see why, let $w \in \mathfrak{M}|_{(\varphi\wedge\psi)}$; by definition, $\mathfrak{M}, w \Vdash \varphi \wedge \psi$. Then, $\mathfrak{M}, w \Vdash \varphi$ and $\mathfrak{M}, w \Vdash \psi$. So, $w \in \mathfrak{M}|_\varphi$ and $\mathfrak{M}|_\varphi, w \Vdash \psi$. This implies $w \in (\mathfrak{M}|_\varphi)|_\psi$; from which we get, $\mathfrak{M}|_{(\varphi\wedge\psi)} \subseteq (\mathfrak{M}|_\varphi)|_\psi$. Similarly, $(\mathfrak{M}|_\varphi)|_\psi \subseteq \mathfrak{M}|_{(\varphi\wedge\psi)}$. Therefore, $(\mathfrak{M}|_\varphi)|_\psi = \mathfrak{M}|_{(\varphi\wedge\psi)}$.

Prop. 6 portrays the relation between the update modality $[\_]$ and CPL-consequence as a simple instance of the notion of dynamic consequence from [3,6].

We introduce some notation to help us deal with pivotal consequence in PML.

Definition 14. *Let (π/χ) be a pivotal rule, and ψ be a pivotal formula; we use $(\pi/\chi)\psi$ as an abbreviation for $\pi \wedge [\chi]\psi$.*

Intuitively, $(\pi/\chi)\psi$ captures the effect of a single P-detachment step in Def. 10 in semantic terms.[6] Prop. 7 gives a more concrete view of how this is done.

Proposition 7. *Let (π/χ) be a pivotal rule, and φ and ψ be CPL-formulas:*

$$\vDash [\varphi](\pi/\chi)\psi \text{ iff } \{\varphi\} \vDash_{\mathsf{CPL}} \pi \text{ and } \{\varphi, \chi\} \vDash_{\mathsf{CPL}} \psi.$$

Proof. Left to right: Suppose that $\vDash [\varphi](\pi/\chi)\psi$. By definition, $\vDash [\varphi](\pi \wedge [\chi]\psi)$. It follows that, $\vDash [\varphi]\pi \wedge [\varphi][\chi]\psi$, and so (a) $\vDash [\varphi]\pi$ and (b) $\vDash [\varphi][\chi]\psi$. We obtain $\{\varphi\} \vDash_{\mathsf{CPL}} \pi$ using (a) and Item 2 in Prop. 6. We obtain $\{\varphi, \chi\} \vDash_{\mathsf{CPL}} \psi$ using (b) and Items 1 and 3 in Prop. 6.

The result in Prop. 7 can be generalized to capture the effect of successive P-detachment steps at the level of pivotal models. This is done in Prop. 8.

[6] Def. 14 allows us to refer indistinctly to a pivotal rule, and to the corresponding formula capturing it. Context will always disambiguate.

Proposition 8. *Let $\bar{\rho} = \rho_1 \dots \rho_n$ be a sequence of pivotal rules s.t. $\rho_i = (\pi_i/\chi_i)$; and φ and ψ be* CPL*-formulas. It follows that $\vDash [\varphi]\bar{\rho}\psi$ is equivalent to*

$$\textit{for all } 1 \leq i \leq n,\ \{\varphi, \chi_1, \dots, \chi_{(i-1)}\} \vDash_{\mathsf{CPL}} \pi_i.$$

Proof. By definition, $[\varphi]\bar{\rho}\psi = [\varphi](\pi_1 \wedge ([\chi_1](\dots(\pi_n \wedge [\chi_n]\psi))))$. The cases where $\bar{\rho}$ is empty or has just one pivotal rule correspond exactly to Item 1 in Prop. 6 and to Prop. 7. The general case makes repeated use of the fact that $\vDash [\gamma](\xi \wedge \eta)$ iff $\vDash [\gamma]\xi \wedge [\gamma]\eta$ and Item 3 in Prop. 7.

It is easy to see that Prop. 8 is the semantic counterpart to Prop. 4. This proposition enables us to *internalize* the effect of P-detachment steps in the language of PML. Moreover, it leads in a natural way to the following semantic formulation of pivotal consequence as a tautology of PML.

Definition 15. *Let P be a set of pivotal rules, and φ and ψ be* CPL*-formulas; define $\{\varphi\} \vDash_P \psi$ iff exists a sequence $\bar{\rho}$ of pivotal rules in P s.t. $\vDash [\varphi]\bar{\rho}\psi$.*

Thm. 2 establishes the parallel between $\vdash_P$ and $\vDash_P$. Its proof is direct from Props. 4 and 8, and Thm. 1.

Theorem 2. *Let P be a set of pivotal rules, and φ and ψ be* CPL*-formulas:*

$$\{\varphi\} \vdash_P \psi \quad \textit{iff} \quad \{\varphi\} \vDash_P \psi.$$

Thm. 2 offers an account of how to capture pivotal consequence semantically, and from a dynamic perspective, as certain tautologies of PML. It may be noted that the definitions and results presented above restrict their attention to singleton sets of premisses. There is no loss of generality in this restriction. Def. 15 can be extended to arbitrary sets of premisses by defining $\varphi = \varphi_1 \wedge \dots \wedge \varphi_n$ for some finite subset $\{\varphi_1 \dots \varphi_n\} \subseteq \Phi$. This reformulation of Def. 15 has no impact on Thm. 2 because of compactness of CPL. We have chosen to deal with singleton sets of premisses to simplify the exposition of the main concepts and results.

Transfer Results. We conclude this section by showing some transfer results from PML to the setting of pivotal consequence. In particular, we tackle *complete axiomatizations*, *expressivity*, and *complexity* of pivotal consequence. These results take advantage of the fact that PML is a modal logic.

Axiom system. Working a logic from a semantic perspective has its advantages. In our case, it enabled us to recast the notion of pivotal rule consequence over CPL formulas into a notion of tautology in PML. At the same time, one may ask whether it is possible to use a syntatic deduction system in PML to determine pivotal rule consequence.

Tab. 1 introduces an axiomatization of PML. This is a standard complete axiomatization for PAL. It includes the S5 axioms and rules for the A modality, and the reduction axioms for eliminating [_] (see, e.g., [11,8] for details).

Theorem 3. *Tab. 1 is a sound and strongly complete axiom system for* PML*.*

Table 1: Axiom system for PML.

Axioms:				
(Taut) CPL-tautologies	(K)	$(\mathsf{A}(\psi \to \varphi) \wedge \mathsf{A}\psi) \to \mathsf{A}\varphi$	(T)	$\mathsf{A}\varphi \to \varphi$
	(4)	$\mathsf{A}\varphi \to \mathsf{AA}\varphi$	(5)	$\neg\mathsf{A}\varphi \to \mathsf{A}\neg\mathsf{A}\varphi$
Rules:	(mp)	from $\varphi \to \psi$ and φ infer ψ	(nec)	from φ infer $\mathsf{A}\varphi$
Reduction Axioms:	(univ)	$[\varphi]\mathsf{A}\psi \leftrightarrow (\varphi \to \mathsf{A}[\varphi]\psi)$	(bot)	$[\varphi]\bot \leftrightarrow (\varphi \to \bot)$
	(imp)	$[\varphi](\psi \to \chi) \to ([\varphi]\psi \to [\varphi]\chi)$	(prop)	$[\varphi]p_i \leftrightarrow (\varphi \to p_i)$

The axiom system for PML is the exact machinery behind single-agent PAL. The global modality A can be seen as a total relation in a Kripke model. Thm. 3 implies that P-deductions can be captured as deductions in single-agent PAL.

Expressive Power. Turning to expressivity, it is important to determine whether two models are structurally equivalent for a particular logic; and to provide characterization results which make coincide the notion of model equivalence and of logical equivalence. In Modal Logic this is related to the notion of bisimulation [2]. Model equivalence in PML is fairly straightforward. It is easily seen that finite models form a Hennessy-Milner class for PML (see, e.g., [10] for details).

Definition 16. *Let $\mathfrak{M}_1, w_1$ and $\mathfrak{M}_2, w_2$ be two pointed pivotal models; we write $\mathfrak{M}_1, w_1 \equiv \mathfrak{M}_2, w_2$ iff $\mathfrak{M}_1, w_1 \Vdash \varphi$ iff $\mathfrak{M}_2, w_2 \Vdash \varphi$ (for all PML-formulas φ).*

Proposition 9 (Hennessy-Milner Property). *The following are equivalent: 1. $\mathfrak{M}_1, w_1 = \mathfrak{M}_2, w_2$; and 2. $\mathfrak{M}_1, w_1 \equiv \mathfrak{M}_2, w_2$.*

Proof. Left to right: trivial. Right to left: we prove $\mathfrak{M}_1, w_1 \neq \mathfrak{M}_2, w_2$ implies $\mathfrak{M}_1, w_1 \not\equiv \mathfrak{M}_2, w_2$. Suppose that $\mathfrak{M}_1, w_1 \neq \mathfrak{M}_2, w_2$; either (1) $w_1 \neq w_2$ or (2) $\mathfrak{M}_1 \neq \mathfrak{M}_2$. From (1), exists $p \in \mathsf{Prop}$ s.t. $w_1(p) \neq w_2(p)$. But then, (w.l.o.g.) $\mathfrak{M}_1, w_1 \Vdash p$ and $\mathfrak{M}_2, w_2 \nVdash p$. Thus, $\mathfrak{M}_1, w_1 \not\equiv \mathfrak{M}_2, w_2$. From (2), (w.l.o.g.) there is $w^* \in \mathfrak{M}_1$ s.t. $w^* \notin \mathfrak{M}_2$. That is, for all $w_i \in \mathfrak{M}_2$, there is p_j s.t. $w^*(p_j) \neq w_i(p_j)$. Moreover, because $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are finite, there is a maximal j such that p_j has this property. Let $\varphi = (l_0 \wedge \ldots \wedge l_j)$ where $l_i = p_i$ if $w^*(p_i) = 1$ and $l_i = \neg p_i$ otherwise. Clearly, $\mathfrak{M}_1, w_1 \Vdash \mathsf{E}\varphi$ (because of w^*), but $\mathfrak{M}_2, w_2 \nVdash \mathsf{E}\varphi$. Thus, $\mathfrak{M}_1, w_1 \not\equiv \mathfrak{M}_2, w_2$.

Prop. 9 shows that finite pivotal models form a Hennessy-Milner class for PML [10]. Intuitively, this means that over this class, model equivalence implies bisimulation – which, for the case of PML, coincides with equality. However, model equivalence is not enough to ensure equality in infinite models. To see why, consider the model $\mathfrak{M}_1 = \{v_i \mid i \in \mathbb{N}, v_i(p_j) = 1 \text{ iff } 0 \leq j \leq i\}$, and $\mathfrak{M}_2 = \mathfrak{M}_1 \cup \{v\}$ with $v(p_j) = 1$ for all $j \in \mathbb{N}$. Clearly $\mathfrak{M}_1 \neq \mathfrak{M}_2$ but when restricted to a finite number of propositional symbols they coincide (i.e., they satisfy the same formulas of PML). Hence $\mathfrak{M}_1, v_1 \equiv \mathfrak{M}_2, v_1$.

Complexity. Finally, we characterize the complexity of checking pivotal consequence by using known complexity results for PAL [13]. To the best of our knowledge, Thm. 4 establishes the exact complexity of pivotal consequence for the first time.

Theorem 4. *Deciding* $\{\varphi\} \vdash_P \psi$ *is* coNP*-complete.*

Proof. coNP-hardness is clear by the coNP-hardness of checking consequence in CPL. Thm. 2 shows that $\{\varphi\} \vdash_P \psi$ is equivalent to $\{\varphi\} \vDash_P \psi$, which (by Def. 15) is the case iff there is a sequence $\bar{\rho}$ of pivotal rules in P s.t. $[\varphi]\bar{\rho}\psi$ is a validity in PML. We know that PML is just a notational variant of single agent PAL. This means that the results of [13] apply to PML. Then, there is a polynomial satisfiability preserving translation (for S5-models) T of formulas from PML into the basic modal language (i.e., formulas without the [_]_ modality). As a consequence of the results in [13], checking satisfiability of a PML-formula is NP-complete. This means that, given $\{\varphi\} \vdash_P \psi$, we can guess a witness $(\bar{\rho}, \mathfrak{M}, w)$ polynomial in (φ, ψ, P) such that $\mathfrak{M}, w \nVdash T([\varphi]\bar{\rho}\psi)$. It is easy to check, in polynomial time, that $(\bar{\rho}, \mathfrak{M}, w)$ is a proper witness:

1. $\bar{\rho}$ should be a sequence, without repetitions, in P^*;
2. $\mathfrak{M}, w$ should be an S5-model that satisfies $\neg T([\varphi]\bar{\rho}\psi)$.

Since satisfiability is NP-complete, deciding $\{\varphi\} \vdash_P \psi$ is coNP-complete.

4.2 Credulous Consequence in PAL

Having dealt with pivotal consequence, we turn our attention to dealing with credulous consequence from a semantic perspective. At this point we fully exploit and take advantage of the expressive power of the modalities of PML.

The key idea is to use the existential modality E to capture the consistency constraint on the use of pivotal rules. We make this precise in Prop. 10 below.

Proposition 10. *For all* $\{\varphi, \psi\} \subseteq \mathsf{Form}_0$*,* $\vDash [\varphi]\mathsf{E}\psi$ *implies* $\{\varphi\} \nvDash_{\mathsf{CPL}} \neg\psi$*.*

It is worth mentioning that the converse of Prop. 10 fails to hold in general. To see why, notice that, e.g., $\nvDash_{\mathsf{CPL}} \neg p$ but $\nvDash [\top]\mathsf{E}p$. That $\nvDash_{\mathsf{CPL}} \neg p$ is obvious. We build a witness for $\nvDash [\top]\mathsf{E}p$ by taking $\mathfrak{M}$ to be a pivotal model comprised of a single $v \in \{0,1\}^{\mathsf{Prop}}$ s.t. $v(p) = 0$. Clearly, $\mathfrak{M}, v \nVdash [\top]\mathsf{E}p$. So, $\nvDash [\top]\mathsf{E}p$. This example brings to the surface why the converse of Prop. 10 may fail to hold in general: pivotal models, when seen as sets of CPL-assignments, do not necessarily include all CPL-counter-models. In the particular case of the example, $\mathfrak{M}$ does not contain at least one CPL-assignment v' s.t. $v'(p) = 1$. The point to be made is: the consistency check that we are after requires us not to miss relevant CPL-assignments. We overcome this issue by restricting our attention to a very particular pivotal model –introduced below.

Definition 17. *Let* $\mathfrak{M}^{\mathsf{CPL}} = \{0,1\}^{\mathsf{Prop}}$*, i.e.,* $\mathfrak{M}^{\mathsf{CPL}}$ *contains all* CPL*-assignments.*

The converse of Prop. 10 holds immediately in $\mathfrak{M}^{\mathsf{CPL}}$.

Proposition 11. *For all* $\{\varphi, \psi\} \subseteq \mathsf{Form}_0$, $\{\varphi\} \not\models_{\mathsf{CPL}} \neg\psi$ *implies* $\mathfrak{M}^{\mathsf{CPL}} \Vdash [\varphi]\mathsf{E}\psi$.

Let us now turn our attention onto how to use Def. 14 to mimic the effect of a single credulous P-detachment step.

Proposition 12. *Let* (π/χ) *be a pivotal rule,* φ *and* ψ *be* CPL*-formulas; then:*

$$\models [\varphi](\pi \wedge \mathsf{E}\chi/\chi)\psi \textit{ implies } \{\varphi\} \models_{\mathsf{CPL}} \pi,\ \{\varphi\} \not\models_{\mathsf{CPL}} \neg\chi, \textit{ and } \{\varphi, \chi\} \models_{\mathsf{CPL}} \psi.$$

Proof. Suppose that $\models [\varphi](\pi \wedge \mathsf{E}\chi/\chi)\psi$. By definition, $\models [\varphi](\pi \wedge \mathsf{E}\chi \wedge [\chi]\psi)$. It follows that: (a) $\models [\varphi]\pi$; (b) $\models [\varphi]\mathsf{E}\chi$; and (c) $\models [\varphi][\chi]\psi$. We obtain $\{\varphi\} \models_{\mathsf{CPL}} \pi$ using (a) and Item 2 in Prop. 6. We obtain $\{\varphi\} \not\models_{\mathsf{CPL}} \neg\chi$ using (b) and Prop. 10. We obtain $\{\varphi, \chi\} \models_{\mathsf{CPL}} \psi$ using (c) and Items 1 and 3 in Prop. 6.

Notice that $\models [\varphi](\pi \wedge \mathsf{E}\chi/\chi)\psi$ does not imply $\{\varphi\} \models \chi$. The condition $\mathsf{E}\chi$ is sufficiently strong to guarantee a consistency check in CPL, but not strong enough to warrant consequence in CPL. Unsurprisingly, as with Prop. 11, the converse of Prop. 12 does not hold in general, but it does over $\mathfrak{M}^{\mathsf{CPL}}$.

Proposition 13. *Let* (π/χ) *be a pivotal rule,* φ *and* ψ *be* CPL*-formulas; then:*

$$\{\varphi\} \models_{\mathsf{CPL}} \pi, \{\varphi\} \not\models_{\mathsf{CPL}} \neg\chi, \textit{ and } \{\varphi, \chi\} \models_{\mathsf{CPL}} \psi \quad \textit{iff} \quad \mathfrak{M}^{\mathsf{CPL}} \Vdash [\varphi](\pi \wedge \mathsf{E}\chi/\chi)\psi.$$

Proof. Right to left: Immediate from Prop. 12. Left to right: We prove the contrapositive, i.e., $\mathfrak{M}^{\mathsf{CPL}} \nVdash [\varphi](\pi \wedge \mathsf{E}\chi/\chi)\psi$ implies (a) $\{\varphi\} \not\models_{\mathsf{CPL}} \pi$ or (b) $\{\varphi\} \models_{\mathsf{CPL}} \neg\chi$ or (c) $\{\varphi, \chi\} \not\models_{\mathsf{CPL}} \psi$. Suppose $\mathfrak{M}^{\mathsf{CPL}} \nVdash [\varphi](\pi \wedge \mathsf{E}\chi/\chi)\psi$; by definition $\mathfrak{M}^{\mathsf{CPL}} \nVdash [\varphi](\pi \wedge \mathsf{E}\chi \wedge [\chi]\psi)$. It follows that either (a') $\mathfrak{M}^{\mathsf{CPL}} \nVdash [\varphi]\pi$; or (b') $\mathfrak{M}^{\mathsf{CPL}} \nVdash [\varphi]\mathsf{E}\chi$; or (c') $\mathfrak{M}^{\mathsf{CPL}} \nVdash [\varphi][\chi]\psi$. We obtain (a) and (c) using (a') and (c'), and Prop. 6. We obtain (b) using the contrapositve of Prop. 11.

We have used $(\pi \wedge \mathsf{E}\chi/\chi)\psi$ to capture the effect of a single *credulous* P-detachment step in semantic terms. To be able to compose the effect of successive credulous P-detachment steps at the level of models we revise Prop. 8.

Definition 18. *Let* $\bar{\rho} = \rho_1 \ldots \rho_n$ *be a sequence of pivotal rules, where each* ρ_i *is of the form* (π_i/χ_i)*; define* $\bar{\rho}_c = (\pi_1 \wedge \mathsf{E}\chi_1/\chi_1) \ldots (\pi_n \wedge \mathsf{E}\chi_n/\chi_n)$.

Proposition 14. *Let* $\bar{\rho} = \rho_1 \ldots \rho_n$ *be a sequence of pivotal rules; and* φ, ψ *be* CPL*-formulas. It follows that* $\mathfrak{M}^{\mathsf{CPL}} \Vdash [\varphi]\bar{\rho}_c\psi$ *is equivalent to*

$\{\varphi\} \cup X \models_{\mathsf{CPL}} \psi$, *and*

for all $1 \leq i \leq n$, $\{\varphi\} \cup X_{(i-1)} \models_{\mathsf{CPL}} \pi_i$ *and* $\{\varphi\} \cup X_{(i-1)} \not\models_{\mathsf{CPL}} \neg\chi_i$,

where: $\rho_i = (\pi_i/\chi_i)$*;* $X_j = \{\chi_1, \ldots, \chi_j\}$*; and* $X = X_n$.

Def. 19 allows us to formulate credulous consequence using announcements.

Definition 19. *Let* P *be a set of pivotal rules, and* φ *and* ψ *be* CPL*-formulas; define* $\{\varphi\} \approx_P \psi$ *iff exists a sequence* $\bar{\rho}$ *of pivotal rules in* P *s.t.* $\mathfrak{M}^{\mathsf{CPL}} \Vdash [\varphi]\bar{\rho}_c\psi$.

Thm. 5 establishes the parallel between $\mathrel{|\!\sim}_P$ and $\approx_P$. Its proof follows from Props. 4 and 14 and Thm. 1.

Theorem 5. *Let P be a set of pivotal rules, φ and ψ be* CPL*-formulas; then:*

$$\{\varphi\} \mathrel{|\!\sim}_P \psi \quad \textit{iff} \quad \{\varphi\} \approx_P \psi.$$

Thm. 5 tells us that ψ is a credulous consequence of φ given P iff $[\varphi]\bar{\rho}_c\psi$ holds everywhere in $\mathfrak{M}^{\mathsf{CPL}}$. In this way, Thm. 5 reduces witnessing credulous consequence to a model checking problem in a particular pivotal model.

Transfer Results. We finish this section discussing some transference results that can be obtained by virtue of Thm. 5.

Axiom System. We begin with a discussion on the axiomatization of $\mathrel{|\!\sim}_P$. Given Thm. 5, we would need for PML formulas to be evaluated in $\mathfrak{M}^{\mathsf{CPL}}$. This makes the axiom system in Tab. 1 insufficient. However, it is not a major obstacle. Let us make an abuse of notation and write $\mathfrak{M}^{\mathsf{CPL}}$ for the class of models containing solely $\mathfrak{M}^{\mathsf{CPL}}$. The sole model in $\mathfrak{M}^{\mathsf{CPL}}$ is needed to capture the "consistency check" required in the credulous use of pivotal rules. $\mathfrak{M}^{\mathsf{CPL}}$ is characterized in Def. 20.

Definition 20. *Let $P = \{p_1, \ldots, p_n\}$ and $Q = \{q_1, \ldots, q_m\}$ be finite subsets of* Prop *s.t. $P \cap Q = \emptyset$; define $\mathsf{Val}(P, Q) = \mathsf{E}((p_1 \wedge \cdots \wedge p_n) \wedge (\neg q_1 \wedge \cdots \wedge \neg q_m))$. Moreover, define* VAL *as the set of all $\mathsf{Val}(P, Q)$.*

Theorem 6. *The axioms and rules in Tab. 1 plus* VAL *yield a sound and complete axiom system for* PML *over the class $\mathfrak{M}^{\mathsf{CPL}}$.*

Complexity. Our last result concerns complexity of deciding default consequence. We provide an upper bound, whose proof relies on the model checking problem for modal logic with the particular model $\mathfrak{M}^{\mathsf{CPL}}$.

Similarly to the case of pivotal consequence, we use Thm. 5, i.e., $\{\varphi\} \mathrel{|\!\sim}_P \psi$ is equivalent to $\{\varphi\} \approx_P \psi$, which is the case (from Def. 19) iff there is a sequence $\bar{\rho}$ of pivotal rules in P s.t. $\mathfrak{M}^{\mathsf{CPL}} \Vdash [\varphi]\bar{\rho}_c\psi$. On a first attempt, we could try to use the same strategy employed in Thm. 4, i.e., guess a polynomial witness $(\bar{\rho}, \mathfrak{M}, w)$. However, notice that, for credulous consequence, $\mathfrak{M}$ cannot be an arbitrary S5-model –since we need to check the satisfiability of $[\varphi]\bar{\rho}_c\psi$ in $\mathfrak{M}^{\mathsf{CPL}}$ (of course, we can use $\mathfrak{M}^{\mathsf{CPL}}$ restricted to the signature of $[\varphi]\bar{\rho}_c\psi$, which is finite, but not polynomial in the input). If we guess an arbitrary polynomial S5-model $\mathfrak{M}$, it is unclear how we can check in polynomial time that it is a proper witness (and the results in [12], see below, imply that being able to do so would result in a collapse of the polynomial hierarchy). In spite of these issues, we can directly establish a PSpace upper bound.

Theorem 7. *Deciding $\{\varphi\} \mathrel{|\!\sim}_P \psi$ is in PSpace.*

Proof. We only need to show that it is possible to model check $[\varphi]\bar{\rho}_c\psi$ in $\mathfrak{M}^{\mathsf{CPL}}$, on the fly, using only polynomial space.

Because PSpace = NPSpace we can, non-deterministically, model check $[\varphi]\bar{\rho}_c\psi$ for a particular, arbitrary, $\bar{\rho}$. The following algorithm $A(\varphi)$ checks that an arbitrary PML-formula φ holds in the restriction of $\mathfrak{M}^{\mathsf{CPL}}$ to the signature of φ in polynomial space.

Let φ be a PML-formula, and i be the largest index s.t. p_i appears in φ. Let $v_1, \ldots, v_{2^i}$ be an enumeration of all the possible assignments over $p_1, \ldots, p_i$, and assume that numbers are represented in binary. Let

$$A(\varphi) := \text{for } k = 1 \text{ to } 2^i \text{ do } F(\varphi, k, \{1, \ldots, 2^i\});$$

where $F(\psi, k, S)$ is defined recursively as follows:

1. if $\psi = \bot$ then return False;
2. if $\psi = p_j$ then return $v_k(p_j) = 1$;
3. if $\psi = \alpha \to \beta$ then return not $F(\alpha, k, S)$ or $F(\beta, k, S)$;
4. if $\psi = \mathsf{A}\alpha$ then
   for $j \in S$ do
   if not $F(\alpha, j, S)$ then return False;
   return True;
5. if $\psi = [\alpha]\beta$ then
   if not $F(\alpha, k, S)$ then return True;
   $S' = \{j \mid F(\alpha, j, S)\}$;
   return $F(\beta, k, S')$;

Remark 2. The complexity of reasoning in different non-monotonic logics has been investigated in [12]. In particular, [12] shows that the problem of deciding credulous consequence is Σ_2^P-complete. Such a result is exact; and, thus, it is stricter than ours. Nonetheless, we believe that Thm. 7 gives a simpler proof of a reasonable small upper bound via a transfer result.

5 Final Remarks

We presented a semantic exploration of the concept of PRC proposed in [14,15] using the standard semantic machinery of PAL. In doing this, we connected PRC and PAL. In more detail, we obtained characterization results of PRC and credulous consequence in PAL. These characterization results led to natural axiomatizations and completeness results. Moreover, they led to transference results of expressivity and complexity. Interestingly, in our framework, credulous consequence turns out to be a model checking problem.

Our main results are Thms. 2 and 5. These can be seen as completeness results that show that the semantics of (a clearly defined fragment of) PAL captures PRC and credulous consequence. When seen in this light our work complements the picture presented in [14,15], where these notions are defined as closure operators on sets of formulas.

We consider our work a first step towards studying different notions of non-monotonic consequence from a dynamic logic perspective. The results in [14,15] are not restricted to PRC, they show other ways of building *bridges* from classical to non-monotonic logic. We would also like to explore these other bridges in our setting. Moreover, it would be interesting to generalize the notion of PRC to logics more expressive than CPL, e.g., modal logics, and to study whether or not we need to extend PAL to characterize these logics.

Our work also opens up the door to explore the use of reasoning methods from modal logic, i.e., tableaux systems and model checkers, in the setting of PRC and in Default Logic. Modal logics boast a trove of automated tools and techniques which could now be applied to non-monotonic reasoning.

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