Algebraic Tools for Default Modal Systems

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Abstract

Default Logic refers to a family of non-monotonic formalisms. Traditionally, default logics have been defined and dealt with via syntactic notions of consequence, in general, in propositional logic or first-order logic. Here, we build default logics on modal logics. First, we present these default logics syntactically. Then, we elaborate on an algebraic counterpart. We do the latter by extending the notion of a modal algebra to accommodate for the main elements of default logics: defaults and extensions. Our algebraic treatment of default logics concludes with an algebraic completeness result and a way of comparing default logics borrowing ideas from the concept of bisimulation in modal logic. To our knowledge, our approach is novel. Interestingly, it also lays the groundwork for studying default logics from a dynamic logic perspective.

1 Introduction

Default Logic refers to a family of non-monotonic formalisms with two main capabilities: reasoning with incomplete knowledge, and dealing with contradictory information. The first of these capabilities is handled by so-called defaults. Defaults are non-admissible rules of inference whose conclusions are subject to annulment. Intuitively, defaults handle reasoning with incomplete knowledge by drawing conclusions which complete what is unknown. The second of these capabilities is handled by so-called extensions. Extensions can be understood as sets of formulas closed under the application of defaults. Intuitively, extensions handle reasoning in the presence of contradictory information by exploring consistent alternatives.

The history of Default Logic traces back to Reiter's seminal work [25]. Since then, many variants of Reiter's original ideas have been proposed – with each variant giving rise to a different default logic (see [3] for a comprehensive summary). For the most part, these variants have focused their attention on what is meant by an extension. In particular, the emphasis has been on how different interactions between defaults, and the rules of inference of the underlying proof calculus, concoct different notions of an extension satisfying one or more properties of interest. This treatment of extensions carries with it the definition and analysis of a default logic from a syntactic perspective. However, the other side of the coin is missing. In studying a logic (of any kind), we also wish to address it from a semantic perspective, either via a model theory or via a class of algebras. The semantic perspective yields interesting completeness results, interpolation properties, bisimulations, etc. Semantic considerations on default logics are mostly absent, making it difficult to investigate their logical properties using standard semantic tools.

Our work. Following the tradition in Default Logic, we start with a formulation of default logics over modal logics via deducibility (i.e., syntactical consequence in the proof calculus). We rely on the notion of global deducibility for modal logics [14]. Our formulation of a default logic is parametric, and can be instantiated with any modal system from K to S5 extended with the universal modality [5]. As we will see, the use of the universal modality is not arbitrary but a necessary tool which simplifies the treatment of defaults and extensions in an algebraic setting. For each of our default logics, we make explicit how defaults interact with the rules of inference of the underlying proof calculus. We do this by integrating the use of the former into the notion of deducibility of the latter.

In addition, we explore our default logics from an algebraic perspective. We do this by extending modal algebras to accommodate for defaults and extensions. Modal algebras are Boolean algebras extended with additional operators for modalities, and they make up the algebraic counterpart of modal systems [32, 16]. Defaults and extensions are incorporated into this setting taking Lindenbaum-Tarski constructions as a starting point. Lindenbaum-Tarski constructions act as algebraic canonical models for sets of permisses. We enrich these constructions with an operator to deal with defaults. This operator can be thought of as "updating" the Lindenbaum-Tarski algebra w.r.t. the application of a default. The result of an update is the algebraic counterpart of an extension. Our algebraic treatment of default logics sets the context to obtain an algebraic completeness result. Moreover, it gives us a way of comparing default logics borrowing ideas from the concept of bisimulation in modal logic.

Related work. Our treatment of defaults and extensions enables us to think of default logics as algebraic "model changing" logics; in the sense of, e.g., public announcement logic [24]. In our case, a model update corresponds to the application of a default (a sort of inference step). The idea of updating a model dynamically to represent syntactic steps of inference can be found in several places in the literature on dynamic logics. For instance, the problem of logical omniscience in epistemic logic (see, e.g., [30]) has been thought of as a property to be achieved after the application of a dynamic operation. In [11, 1, 21, 26], omniscience is achieved by updating models containing sets of formulas. In [29, 19] the updates are performed over awareness relational models. Dynamics of evidence are presented in [28, 31] over neighbourhood models. Finally, dynamic modalities allowing to achieve introspective states over Kripke models are introduced in [12, 13].

Closer to our work is the algebraic treatment of public announcements introduced in [23]. Therein, the algebraic submodel relation induced by the announcement of a formula ψ is represented by taking the quotient algebra modulo an equivalence relation given by ψ. We show that the application of a default δ can be captured in a similar way, i.e., by taking the quotient algebra modulo the equivalence relation given by the new knowledge added by δ. We elaborate on this idea in Sec. 4.

Motivation. Our choice of building default logics on modal logics is grounded on the fact that modal logics provide a wide spectrum of logics which are more expressive than propositional logic, but which also have better computational properties than first-order logic. Modal logics also have a well-developed algebraic theory in terms of modal algebras. In our constructions we exploit the combination of these two features. As it turns out, defaults are better modeled by means of a global consequence relation, which we capture internally using the universal modality. While we do not pursue it here, default logics built on modal logics is interesting from the perspective of applying the developed formalism to particular scenarios. This is particularly true in the setting of description logics – where it is possible to think of defaults as a way of capturing exceptions to a taxonomy of concepts modeled in a knowledge base (see [4]).

Main contributions. We offer a syntactic and an algebraic treatment of default logics built over modal logics and study their properties. Syntactically, the construction of a default logic over a modal logic results in what we refer to as default modal system. These default sysmtes are parametric on a modal system and a set of defaults. We show how defaults interact with the rules of inference of the underlying modal system by providing a suitable notion of deduction by default. Algebraically, we recast defaults and extensions in the setting of modal algebras. This enables us to obtain a completeness result for default modal systems using standard algebraic tools. Dealing with defaults and extensions in the setting of modal algebras opens the door to the study metalogical properties of default modal systems from a algebraic perspective, and can be seen as a first step towards an algebraization of default logics.

The contributions thus far mentioned extend the ideas and results introduced in [9]. We include examples to illustrate and clarify some important concepts and definitions. We also provide detailed proofs of results. Examples and detailed proofs had been ommitted from [9] due to reasons of space. As a completely novel result, we present a notion of bisimulation which allows to characterize the expressivity of a default modal system. We consider this result to be a substantial contribution. Our notion of bisimulation builds on the notion of bisimulation of the underlying modal system. Bisimulations are the de-facto way of proving semantic equivalences in Modal Logic. Our results show that bisimulations can also be used to compare default modal systems. We elaborate on a application of these results to the problem of equivalence of default theories. Our notion of bisimulation gains interest since, as it is usual with bisimulations, we only need to inspect properties that are relative to particular points. This in contrast to other procedures used to compare default theories which examine complete entities (see e.g., [27, 20]).

We conclude by discussing how recasting defaults and extensions in the setting of modal algebras lays the groundwork to study default systems from a dynamic logic perspective.

Structure of the article. Sec. 2 covers background material, in particular concerning Boolean algebras, modal systems, and the algebraization of modal systems. Sec. 3 contains our main results. Sec. 3.1 introduces default modal systems. Sec. 3.2 presents default deducibility. Sec. 3.3 provides our algebraic characterization of defaults and extensions, and a completeness theorem. Sec. 3.4 contains novel ideas and results. Therein, we study some properties of default systems using bisimulations, and discuss an application for comparing so-called default theories. In Sec. 4 we discuss default modal systems from a dynamic logic perspective. Sec. 5 offers some final remarks.

2 Background

2.1 Boolean Algebra in a Nutshell

We briefly introduce the basics of Boolean algebras (see, e.g., [17] for details).

Definition 2.1. A *Boolean Algebra* (BA) is a structure $\mathbf{A} = \langle A, *, -, 1\rangle$ of type 2−1−0 satisfying a well-known set of equations. The set A is also denoted as $|\mathbf{A}|$. We consider operations $+$ and 0 defined as $a + b = -(-a * -b)$, and $0 = -1$.

Definition 2.2. Every BA $\mathbf{A}$ is equipped with a partial order $\preceq_{\mathbf{A}}$ defined as $x \preceq_{\mathbf{A}} y$ iff $x = x * y$ (sometimes we omit the subindex $\mathbf{A}$ and write just $\preceq$). We write $\uparrow X = \{\, y \mid \text{there is } x \in X \text{ s.t. } x \preceq y \,\}$. A filter is a non-empty subset $F \subseteq |\mathbf{A}|$ s.t.: $F = \uparrow F$ and for all $x, y \in F$, $(x * y) \in F$. A filter is principal if it is of the form $\uparrow\{a\}$ for $a \in |\mathbf{A}|$. A filter F is *proper* if $0 \notin F$.

2.2 Modal Systems

We begin by making precise the set Form of well formed formulas we work with.

Definition 2.3. Let $\mathsf{Prop} = \{\, p_i \mid i \in \mathbb{N} \,\}$ be a denumerable set of *proposition symbols*; the set Form of well formed formulas (wffs, or simply formulas) is determined by the grammar

$$\varphi, \psi ::= p_i \mid \top \mid \neg\varphi \mid \varphi \wedge \psi \mid \Box\varphi \mid \boxed{\mathsf{u}}\varphi.$$

We use the usual abbreviations: $\bot$, $\varphi \vee \psi$, $\varphi \to \psi$, $\varphi \leftrightarrow \psi$, $\Diamond\varphi$ and $\Diamond_{\mathsf{u}}\varphi$.

The set Form can be seen as an enrichment of the basic modal language with the universal modality $\boxed{\mathsf{u}}$. We use the universal modality as a technical tool to internalize a global consequence relation.

By a modal system we mean any of those arising from Def. 2.4.

Definition 2.4. A modal system is fully determined by a set of wffs called axioms, and the rules of inference of *modus ponens* (mp) and *universal generalization* (u) below:

$$\frac{\varphi \quad \varphi \to \psi}{\psi}\ (\mathsf{mp}) \qquad\qquad \frac{\varphi}{\boxed{\mathsf{u}}\varphi}\ (\mathsf{u}).$$

The smallest set of axioms we consider consists solely of all instances of *propositional tautologies* and all instances of the *schemas*:

1. $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$;
2. $\boxed{\mathsf{u}}(\varphi \to \psi) \to (\boxed{\mathsf{u}}\varphi \to \boxed{\mathsf{u}}\psi)$;
3. $\boxed{\mathsf{u}}\varphi \to \varphi$;
4. $\varphi \to \boxed{\mathsf{u}}\Diamond_{\mathsf{u}}\varphi$;
5. $\boxed{\mathsf{u}}\varphi \to \boxed{\mathsf{u}}\boxed{\mathsf{u}}\varphi$;
6. $\boxed{\mathsf{u}}\varphi \to \Box\varphi$.

The previous axioms and rules of inference give rise to the modal system $\mathsf{K}^{\boxed{\mathsf{u}}}$. The modal system $\mathsf{K}^{\boxed{\mathsf{u}}}$ is the most basic modal system we will consider. The rest of the modal systems we consider are obtained from it by adding as additional axioms all instances of any of the schemas below, or any combination thereof:

(4) $\Box\varphi \to \Box\Box\varphi$ (5) $\Diamond\varphi \to \Box\Diamond\varphi$ (B) $\varphi \to \Box\Diamond\varphi$ (D) $\Box\varphi \to \Diamond\varphi$ (T) $\Box\varphi \to \varphi$

E.g., the modal system $\mathsf{D}^{\boxed{\mathsf{u}}}$ has as axioms all axioms of $\mathsf{K}^{\boxed{\mathsf{u}}}$, plus all instances of the schema D. Similarly, the modal systems $\mathsf{S4}^{\boxed{\mathsf{u}}}$ and $\mathsf{S5}^{\boxed{\mathsf{u}}}$ have as axioms those of $\mathsf{K}^{\boxed{\mathsf{u}}}$, plus all instances of the schemas T and 4, and T and 5, respectively.

For each modal system, we define a deducibility, i.e., syntactic consequence, relation between sets of formulas and formulas. This relation is given in Def. 2.5.

Definition 2.5. Let $\mathsf{M}^{\boxed{\mathsf{u}}}$ be a modal system; an $\mathsf{M}^{\boxed{\mathsf{u}}}$-*deduction of* φ *from* Φ is a finite sequence $\psi_1 \dots \psi_n$ of formulas such that $\psi_n = \varphi$, and for each $k < n$ at least one of the following conditions hold:

1. ψ_k is an axiom of $\mathsf{M}^{\boxed{\mathsf{u}}}$;
2. ψ_k is a premiss, i.e., $\psi_k \in \Phi$;
3. ψ_k is obtained using mp, i.e., there are $i, j < k$ s.t. $\psi_j = \psi_i \to \psi_k$;
4. ψ_k is obtained using u, i.e., there is $j < k$ s.t. $\psi_k = \boxed{\mathsf{u}}\psi_j$.

We write $\Phi \vdash_{\mathsf{M}^{\boxed{\mathsf{u}}}} \varphi$ iff there is an $\mathsf{M}^{\boxed{\mathsf{u}}}$-deduction of φ from Φ. The relation $\vdash_{\mathsf{M}^{\boxed{\mathsf{u}}}}$ is commonly referred to as *global*.

We end this section by taking note of the following properties of $\vdash_{\mathsf{M}^{\boxed{\mathsf{u}}}}$.

Proposition 2.1. The following properties hold:

1. If $\vdash_{\mathsf{M}^{\boxed{\mathsf{u}}}} \varphi$, then, $\vdash_{\mathsf{M}^{\boxed{\mathsf{u}}}} \Box\varphi$.
2. If $\Phi \cup \{\varphi\} \vdash_{\mathsf{M}^{\boxed{\mathsf{u}}}} \psi$, then, $\Phi \vdash_{\mathsf{M}^{\boxed{\mathsf{u}}}} \boxed{\mathsf{u}}\varphi \to \psi$.

The first item in Prop. 2.1 refers to the *necessitation* property in modal logics, whereas the second item refers to a version of the *deduction theorem*.

2.3 Algebraizing Modal Systems

We present the semantics of a modal system from an algebraic perspective. More precisely, we associate with each modal system a class of algebras in a way such that the properties of the modal system are in correspondence to the properties of the class of algebras. The algebraic treatment of modal systems is instrumental to perform default reasoning from a semantic point of view, and to viewing default reasoning as a logic of *updates*. We elaborate on these ideas later on. For now, we focus on introducing some basic concepts and results of the classes of algebras we will work with. To this end, we follow closely [32], and borrow ideas and results from [16, 18].

Definition 2.6. The *formula algebra* associated to the set Form of formulas is the structure $\mathbf{F} = \langle \mathsf{Form}, \wedge, \neg, \top, \Box, \boxed{\mathsf{u}} \rangle$ where: $\neg$, $\Box$, $\boxed{\mathsf{u}}$ are seen as unary operations, and $\wedge$ is seen as a binary operation. These operations are defined in the ovbious way, i.e.: $\neg$ applied to $\varphi \in \mathsf{Form}$ returns $\neg\varphi \in \mathsf{Form}$; $\Box$ applied to $\varphi \in \mathsf{Form}$ returns $\Box\varphi \in \mathsf{Form}$; $\boxed{\mathsf{u}}$ applied to $\varphi \in \mathsf{Form}$ returns $\boxed{\mathsf{u}}\varphi \in \mathsf{Form}$; and $\wedge$ applied to $\varphi, \psi \in \mathsf{Form}$ returns $\varphi \wedge \psi \in \mathsf{Form}$.

The standard algebraic treatment of Classical Propositional Logic refers to Boolean algebras (as interpretation structures) and filters (as the semantic counterpart of deducibility). The analogous concepts for the case of modal systems are so-called $\boxed{\sqcup}$-modal algebras, and *open filters*, respectively.

Definition 2.7. A $\boxed{\sqcup}$*-modal algebra* is a structure $\mathbf{M} = \langle B, *, -, 1, f^{\Box}, f^{\boxed{\sqcup}} \rangle$ where: $\langle B, *, -, 1\rangle$ is a Boolean algebra; and $f^{\Box}$ and $f^{\boxed{\sqcup}}$ are unary operations satisfying the following set of equations

$$\begin{array}{rcl} f^{\Box}(1) = 1 & \qquad & f^{\boxed{\sqcup}}(b_1) \preccurlyeq b_1 \\ f^{\Box}(b_1 * b_2) = f^{\Box}(b_1) * f^{\Box}(b_2) & & f^{\boxed{\sqcup}}(b_1) \preccurlyeq f^{\boxed{\sqcup}}(-f^{\boxed{\sqcup}}(-b_1)) \\ f^{\boxed{\sqcup}}(1) = 1 & & f^{\boxed{\sqcup}}(b_1) \preccurlyeq f^{\boxed{\sqcup}} f^{\boxed{\sqcup}}(b_1) \\ f^{\boxed{\sqcup}}(b_1 * b_2) = f^{\boxed{\sqcup}}(b_1) * f^{\boxed{\sqcup}}(b_2) & & f^{\boxed{\sqcup}}(b_1) \preccurlyeq f^{\Box}(b_1). \end{array}$$

An *open filter* in $\mathbf{M}$ is a subset $F \subseteq B$ such that: F is a filter in $\langle B, *, -, 1\rangle$, and for all $b \in F$, $f^{\boxed{\sqcup}}(b) \in F$.

Definition 2.8. An *interpretation* of the formula algebra $\mathbf{F}$ on a $\boxed{\sqcup}$-modal algebra $\mathbf{M} = \langle B, *, -, 1, f^{\Box}, f^{\boxed{\sqcup}}\rangle$, a.k.a. an interpretation on $\mathbf{M}$, is a homomorphism $v : \mathbf{F} \to \mathbf{M}$ such that:

$$\begin{array}{lll} v(\top) = 1 \qquad & v(\neg\varphi) = -v(\varphi) \qquad & v(\Box\varphi) = f^{\Box}(v(\varphi)) \\ & v(\varphi \wedge \psi) = v(\varphi) * v(\psi) & v(\boxed{\sqcup}\varphi) = f^{\boxed{\sqcup}}(v(\varphi)). \end{array}$$

Proposition 2.2. Every interpretation v on $\mathbf{M}$ is uniquely determined by an assignment $v_0 : \mathsf{Prop} \to |\mathbf{M}|$.

Definition 2.9. Let $\mathbf{M}$ be a $\boxed{\sqcup}$-modal algebra; we define:

1. an *equation* as a pair (φ, ψ) of formulas; written as $\varphi \approx \psi$;
2. an equation $\varphi \approx \psi$ is *valid under an interpretation* v *on* $\mathbf{M}$ iff $v(\varphi) = v(\psi)$;
3. an equation is *valid in* $\mathbf{M}$ iff it is valid under all interpretations on $\mathbf{M}$.

We write $\mathbf{M}, v \vDash \varphi \approx \psi$ iff the equation $\varphi \approx \psi$ is valid under v; and $\mathbf{M} \vDash \varphi \approx \psi$ if the equation $\varphi \approx \psi$ is valid in $\mathbf{M}$.

We are now in a position to connect $\boxed{\sqcup}$-modal algebras and modal systems.

Proposition 2.3. Let $\mathsf{M}^{\boxed{\sqcup}}$ be a modal system, and $\Phi \cup \{\varphi, \psi\}$ a set of formulas; define $\varphi \cong^{\Phi}_{\mathsf{M}^{\boxed{\sqcup}}} \psi$ iff $\Phi \vdash_{\mathsf{M}^{\boxed{\sqcup}}} \varphi \leftrightarrow \psi$. The relation $\cong^{\Phi}_{\mathsf{M}^{\boxed{\sqcup}}}$ is a congruence on $\mathbf{F}$.

Definition 2.10. Let $\mathsf{M}^{\boxed{\sqcup}}$ be a modal system, and Φ be a set of formulas; the $\mathsf{M}^{\boxed{\sqcup}}$*-Lindenbaum-Tarski algebra*, or $\mathsf{M}^{\boxed{\sqcup}}$*-LT algebra*, of Φ is the structure $\mathbf{L}^{\Phi}_{\mathsf{M}^{\boxed{\sqcup}}} = \langle \mathsf{Form}/_{\cong^{\Phi}_{\mathsf{M}^{\boxed{\sqcup}}}}, *_{\cong^{\Phi}_{\mathsf{M}^{\boxed{\sqcup}}}}, -_{\cong^{\Phi}_{\mathsf{M}^{\boxed{\sqcup}}}}, 1_{\cong^{\Phi}_{\mathsf{M}^{\boxed{\sqcup}}}}, f^{\Box}_{\cong^{\Phi}_{\mathsf{M}^{\boxed{\sqcup}}}}, f^{\boxed{\sqcup}}_{\cong^{\Phi}_{\mathsf{M}^{\boxed{\sqcup}}}} \rangle$ where:

$$\begin{array}{lll} 1_{\cong^{\Phi}_{\mathsf{M}^{\boxed{\sqcup}}}} = [\top]_{\cong^{\Phi}_{\mathsf{M}^{\boxed{\sqcup}}}} & -_{\cong^{\Phi}_{\mathsf{M}^{\boxed{\sqcup}}}}([\varphi]_{\cong^{\Phi}_{\mathsf{M}^{\boxed{\sqcup}}}}) = [\neg\varphi]_{\cong^{\Phi}_{\mathsf{M}^{\boxed{\sqcup}}}} & f^{\Box}_{\cong^{\Phi}_{\mathsf{M}^{\boxed{\sqcup}}}}([\varphi]_{\cong^{\Phi}_{\mathsf{M}^{\boxed{\sqcup}}}}) = [\Box\varphi]_{\cong^{\Phi}_{\mathsf{M}^{\boxed{\sqcup}}}} \\ & [\varphi]_{\cong^{\Phi}_{\mathsf{M}^{\boxed{\sqcup}}}} *_{\cong^{\Phi}_{\mathsf{M}^{\boxed{\sqcup}}}} [\psi]_{\cong^{\Phi}_{\mathsf{M}^{\boxed{\sqcup}}}} = [\varphi \wedge \psi]_{\cong^{\Phi}_{\mathsf{M}^{\boxed{\sqcup}}}} & f^{\boxed{\sqcup}}_{\cong^{\Phi}_{\mathsf{M}^{\boxed{\sqcup}}}}([\varphi]_{\cong^{\Phi}_{\mathsf{M}^{\boxed{\sqcup}}}}) = [\boxed{\sqcup}\varphi]_{\cong^{\Phi}_{\mathsf{M}^{\boxed{\sqcup}}}}. \end{array}$$

The *canonical interpretation* v on $\mathbf{L}^{\Phi}_{\mathsf{M}^{\boxed{\sqcup}}}$ is defined as $v(\varphi) = [\varphi]_{\cong^{\Phi}_{\mathsf{M}^{\boxed{\sqcup}}}}$.

Proposition 2.4. Every $\mathsf{M}^{\boxed{\sqcup}}$-Lindenbaum-Tarski algebra is a $\boxed{\sqcup}$-modal algebra.

Theorem 2.1. For every modal system $\mathsf{M}^{\boxed{\sqcup}}$, $\Phi \vdash_{\mathsf{M}^{\boxed{\sqcup}}} \varphi$ iff $\mathbf{L}^{\Phi}_{\mathsf{M}^{\boxed{\sqcup}}} \vDash \varphi \approx \top$.

The algebraic completeness of a modal system $\mathsf{M}^{\boxdot}$ w.r.t. a corresponding subclass of $\boxdot$-modal algebras is obtained as a corollary of Thm. 2.1 by showing that the $\mathsf{M}^{\boxdot}$-LT $\boxdot$-modal algebra belongs to the subclass. In this way, $\mathsf{M}^{\boxdot}$-LT $\boxdot$-modal algebras act as "algebraic canonical models" for sets of formulas, i.e., they provide witnesses for $\Phi \nvdash_{\mathsf{M}^{\boxdot}} \varphi$. We make full use of, and benefit from, $\mathsf{M}^{\boxdot}$-LT $\boxdot$-modal algebras in Sec. 3.3.

3 Default Modal Logic

In this section we integrate the main elements of Default Logic, *defaults* and *extensions*, into modal systems. This integration yields what we call a *default modal system*. For each default modal system, we introduce an associated notion of *deduction by default*, which shows how defaults interact with the notion of deduction for the underlying modal system. Moreover, we present how a default modal system can be viewed from an algebraic perspective, and prove a completeness result using algebraic tools. We also show how the algebraic setting for default modal systems offers a natural way of comparing default logics borrowing ideas from the concept of a bisimulation in modal logic. As some final remarks, we discuss how the algebraic treatment of default modal systems can be seen as an update operation on algebraic structures. This opens up the door to thinking about default systems from a dynamic logic perspective (akin to public announcements).

Remark 1. To avoid cluttering the notation with subscripts, in what follows, we assume that $\mathsf{M}^{\boxdot}$ is an arbitrary but fixed modal system and use $\vdash$ for $\vdash_{\mathsf{M}^{\boxdot}}$.

3.1 Default Modal Systems

We start by introducing defaults and extensions in Defs. 3.1 and 3.2, respectively. These definitions are adapted from [25].

Definition 3.1. A default is a triple (π, ρ, χ) of formulas written as $\pi : \rho \,/\, \chi$. The formulas π, ρ, and χ, are called prerequisite, justification, and consequent.

Definition 3.2. Let Φ be a set of formulas and Δ a set of defaults; define a function $\mathsf{D}^{\Phi}_{\Delta}$ s.t. for all sets of formulas Ψ, $\mathsf{D}^{\Phi}_{\Delta}(\Psi)$ is the $\subseteq$-smallest set of formulas which satisfies:

(a) $\Phi \subseteq \mathsf{D}^{\Phi}_{\Delta}(\Psi)$;

(b) $\mathsf{D}^{\Phi}_{\Delta}(\Psi) = \{\, \psi \mid \mathsf{D}^{\Phi}_{\Delta}(\Psi) \vdash_{\mathsf{M}^{\boxdot}} \psi \,\}$;

(c) for all $\pi : \rho \,/\, \chi \in \Delta$, if $\pi \in \mathsf{D}^{\Phi}_{\Delta}(\Psi)$ and $\neg\rho \notin \Psi$, then, $\chi \in \mathsf{D}^{\Phi}_{\Delta}(\Psi)$.

A set E of formulas is an *extension* of Φ under Δ iff it is a fixed point of $\mathsf{D}^{\Phi}_{\Delta}$, i.e., $\mathrm{E} = \mathsf{D}^{\Phi}_{\Delta}(\mathrm{E})$. We use $\mathsf{E}^{\Phi}_{\Delta}$ to indicate the set of all extensions of Φ under Δ.

In the literature on Default Logic, defaults are intuitively understood as defeasible rules of inference, i.e., rules of inference whose conclusions are subject to annulment, or rules which allow us to "jump" to conclusions. In turn, extensions can be thought of akin to theories generated by a set of formulas. In this light, an extension is a set of formulas containing Φ, closed under $\vdash$, and saturated under the application of the defaults in Δ. The next two examples illustrate two properties of extensions: multiplicity and absence of extensions.

Example 1. In the context of the modal system $\mathsf{K}^{\boxdot}$, consider sets $\Phi = \{\Diamond p\}$ and $\Delta = \{\Diamond p : \Diamond\neg p \ / \ \Diamond\neg p, \Diamond p : \Box p \ / \ \Box p\}$; the set $\mathsf{E}^{\Phi}_{\Delta}$ of extensions of Φ under Δ consists of exactly two extensions: (1) the set $\mathrm{E}_1 = \{\, \varphi \mid \{\Diamond p, \Diamond\neg p\} \vdash_{\mathsf{K}^{\boxdot}} \varphi \,\}$; and (2) the set $\mathrm{E}_2 = \{\, \varphi \mid \{\Diamond p, \Box p\} \vdash_{\mathsf{K}^{\boxdot}} \varphi \,\}$.

Each of the extensions in Ex. 1 corresponds to the application of each default in Δ. Once one default has been applied, the application of the other one is blocked. This example illustrates how to handle contradictory information in Default Logic, i.e., via consistent alternatives.

Example 2. In the context of the modal system $\mathsf{K}^{\boxdot}$, consider sets $\Phi = \{\Diamond p\}$ and $\Delta = \{\Diamond p : \Diamond q \ / \ \Box\neg q\}$; the set $\mathsf{E}^{\Phi}_{\Delta}$ of extensions of Φ under Δ is empty, i.e., $\mathsf{E}^{\Phi}_{\Delta} = \emptyset$, i.e., there are no extensions of Φ under Δ.

Ex. 2 highlights a subtletly in thinking of extensions as being constructed by the successive application of defaults: applying a default may result in its own annulment. To make this point clear, w.l.o.g., notice that plausible candidates for extensions are: the set $\mathrm{E}_1 = \{\, \varphi \mid \{\Diamond p\} \vdash_{\mathsf{K}^{\boxdot}} \varphi \,\}$ (i.e., not applying the default); or the set $\mathrm{E}_2 = \{\, \varphi \mid \{\Diamond p, \Box\neg q\} \vdash_{\mathsf{K}^{\boxdot}} \varphi \,\}$ (i.e., applying the default). It can easily be verified that neither of these sets is a fixed point of $\mathsf{D}^{\Phi}_{\Delta}$. More precisely, $\mathsf{D}^{\Phi}_{\Delta}(\mathrm{E}_1) = \mathrm{E}_2$ and $\mathsf{D}^{\Phi}_{\Delta}(\mathrm{E}_2) = \mathrm{E}_1$. This results in $\mathsf{E}^{\Phi}_{\Delta} = \emptyset$.

The definition of a default modal system arises as a natural construction over a modal system by incorporating defaults and extensions.

Definition 3.3. A default modal system is a tuple $\Delta\mathsf{M}^{\boxdot} = \langle \mathsf{M}^{\boxdot}, \Delta, \mathsf{E} \rangle$ where: $\mathsf{M}^{\boxdot}$ is a modal system, Δ is a set of defaults, and E is a function s.t. for all sets Φ of formulas returns $\mathsf{E}^{\Phi}_{\Delta}$.

In analogy with the case in modal systems, we associate with each default modal system a relation $\mathrel{|\!\sim}$ between sets of formulas and formulas. This relation is based on the relation $\vdash$ and it can be understood as its default version. This is made clear in Def. 3.4.

Definition 3.4. Let $\Delta\mathsf{M}^{\boxdot}$ be a default modal system; define

$$\Phi \mathrel{|\!\sim}_{\Delta\mathsf{M}^{\boxdot}} \varphi \quad \text{iff} \quad \varphi \in \mathrm{E} \text{ for some } \mathrm{E} \in \mathsf{E}^{\Phi}_{\Delta}.$$

We drop the subscript $\Delta\mathsf{M}^{\boxdot}$ when it can be understood from the context. We use $\mathrel{|\!\sim} \varphi$ as a shorthand for $\emptyset \mathrel{|\!\sim} \varphi$.

The relation $\mathrel{|\!\sim}$ is called *credulous* in the literature on Default Logic. This name is due to the fact that the existence of just one extension is enough to grant the inference (see [3]). The principle of *monotonicity* fails for $\mathrel{|\!\sim}$. In other words: it is not necessarily the case that if $\Phi \mathrel{|\!\sim} \varphi$, then $\Phi \cup \Psi \mathrel{|\!\sim} \varphi$.

Given that $\mathrel{|\!\sim}$ is built on $\vdash$, we may ask ourselves which properties of $\vdash$ are preserved by $\mathrel{|\!\sim}$. This question does not have an obvious answer, e.g., monotonicity is already not preserved. To this end, we introduce Def. 3.5 as a basis on which to start properly frame this question.

Definition 3.5. The relation $\mathrel{|\!\sim}$ interprets $\vdash$ iff if $\Phi \vdash \varphi$ then $\Phi \mathrel{|\!\sim} \varphi$.

Interpretability seems to be a natural requirement on $\mathrel{|\!\sim}$. However, as established in Ex. 2 (which shows that sometimes extensions do not exist) this property fails to hold in general. To overcome this problem we can go down two possible paths: (i) modify Def. 3.2 to guarantee the existence of extensions;

or (ii) single out defaults for which extensions are guaranteed to exist. Among the most popular modifications of Def. 3.2 which guarantee the existence of extensions we have: *justified* extensions (see [22]); and *constrained* extensions (see [10]). For option (ii), we have the set of *well-behaved*[1] defaults as a very large and natural set which guarantees the existence of extensions (see [25]). Going down path (i) overburdens the definition of an extension with additional machinery which departs from the purposes of our work here. For this reason, we choose to go down path (ii); i.e., we restrict ourselves to well-behaved defaults. Interestingly enough, extensions, justified extensions, and constrained extensions, coincide for well-behaved defaults (see [15, 8]).

Definition 3.6. A default $\pi : \rho \,/\, \chi$ is well-behaved iff $\rho = \chi$. We use π/χ as notation for well-behaved defaults. A set of defaults Δ is well-behaved iff all defaults in Δ are well-behaved. A default modal system is well-behaved iff its set of defaults is well-behaved.

Proposition 3.1. In every well-behaved default modal system, $\mathrel{|\!\sim}$ interprets $\vdash$.

Proof. Notice that $\Phi \subseteq \mathrm{E}$ for all $\mathrm{E} \in \mathsf{E}_\Delta^\Phi$. The result follows immediately from this and the fact that extensions are guaranteed to exist.

We conclude this section by drawing attention to an interesting point regarding necessitation in default modal systems in Prop. 3.2 (cf. item 1 in Prop. 2.1).

Proposition 3.2. In any default modal system, if $\mathrel{|\!\sim} \varphi$, then $\mathrel{|\!\sim} \Box\varphi$.

Proof. Suppose that $\mathrel{|\!\sim} \varphi$; by definition, there is an $\mathrm{E} \in \mathsf{E}_\Delta^\Phi$ s.t. $\mathrm{E} \vdash \varphi$. It follows that $\mathrm{E} \vdash \Box\varphi$. Thus, $\mathrel{|\!\sim} \Box\varphi$.

Prop. 3.2 shows that *necessitation* is preserved by $\mathrel{|\!\sim}$. In turn, we may wonder whether it is possible to obtain the form of the deduction theorem in Prop. 2.1 for $\mathrel{|\!\sim}$; i.e., whether if $\Phi \cup \{\varphi\} \mathrel{|\!\sim} \psi$, then, $\Phi \mathrel{|\!\sim} \boxdot\varphi \to \psi$. Unfortunately, as the next example shows, this property fails to hold for an arbitrary default modal systems (even in the presense of $\boxdot$).

Example 3. In the context of the modal system $\mathsf{K}^{\boxdot}$, consider sets $\Phi = \{p\}$ and $\Delta = \{p/\Diamond p\}$; then, $\mathsf{E}_\Delta^\Phi = \{\{p, \Diamond p\}\}$ and $\mathsf{E}_\Delta^\emptyset = \{\emptyset\}$. Clearly, $\{p, \Diamond p\} \vdash_{\mathsf{K}^{\boxdot}} \Diamond p$ and $\nvdash_{\mathsf{K}^{\boxdot}} \boxdot p \to \Diamond p$. This means that $\{p\} \mathrel{|\!\sim}_{\Delta\mathsf{K}^{\boxdot}} \Diamond p$ and also that $\not\mathrel{|\!\sim}_{\Delta\mathsf{K}^{\boxdot}} \boxdot p \to \Diamond p$.

3.2 Deducibility in Default Modal Systems

We formulate a notion of *deduction by default*, or *default deduction*, for an arbitrary but fixed well-behaved default modal system. This notion of a deduction by default extends that of a deduction by incorporating defaults in a natural way.

Definition 3.7. A *deduction by default*, or *default deduction*, of φ from Φ is a finite sequence $\psi_1 \ldots \psi_n$ of formulas s.t. $\psi_n = \varphi$, and for each $k < n$ at least one of the following conditions hold:

1. ψ_k is a theorem of $\vdash$, i.e., $\vdash \psi_k$;
2. ψ_k is a premiss, i.e., $\psi_k \in \Phi$;

[1] In the literature on Default Logic well-behaved defaults are called normal. We avoid using this terminology here to avoid any confusion with normality in Modal Logic.

3. ψ_k is obtained using mp, i.e., there are $i, j < k$ s.t. $\psi_j = \psi_i \to \psi_k$;
4. ψ_k is obtained using u, i.e., there is $j < k$ s.t. $\psi_k = \boxdot\psi_j$;
5. ψ_k is obtained using Δ-detachment, i.e., there is $j < k$ s.t. $\psi_j/\psi_k \in \Delta$.

A default deduction is *credulous* whenever:

$$(\Phi \cup \{\, \psi_i \mid 1 \leq i \leq n \,\}) \vdash \bot \quad \text{iff} \quad \Phi \vdash \bot. \tag{1}$$

We write $\Phi \vdash^* \varphi$ iff there is a credulous default deduction of φ from Φ.

The notion of a credulous default deduction extends the notion of deduction in the underlying modal system with a rule of default detachment and the condition of being credulous. The rule of default detachment enables us to introduce defaults in the reasoning task and shows us how defaults interact with the rules of the underlying proof system. The condition of being credulous in Eq. (1) captures the fact that defaults cannot be a source of inconsistency. Intuitively, a credulous default deduction of φ from Φ internalizes the construction of (part of) an extension containing φ together with the deduction which witnesses this containment. This is made precise in the following result.

Theorem 3.1. For any set of formulas $\Phi \cup \{\varphi\}$, $\Phi \vdash^* \varphi$ iff $\Phi \vdash \varphi$.

Proof. W.l.o.g. we prove the result for $\Phi \nvdash \bot$. To simplify the proof, we use an alternative characterization of extensions in terms of *closed generating sequences* (which adapts a definition of a closed process in [2]).

By a Δ-sequence we mean a (potentially infinite) sequence $\bar{\delta} = \delta_1\delta_2\delta_3\ldots$ of defaults of Δ. The following notation is useful: (a) $\bar{\delta}|_n = \delta_1 \ldots \delta_n$; (b) $\delta_i = \pi_i/\chi_i$; and (c) $\mathrm{X}_{\bar{\delta}} = \{\, \chi_i \mid \pi_i/\chi_i \in \bar{\delta} \,\}$. A Δ-sequence $\bar{\delta}$ is called *generating* iff for all indices i of $\bar{\delta}$: (d) $\Phi \cup \mathrm{X}_{(\bar{\delta}|_{(i-1)})} \vdash \pi_i$; and (e) $\big(\Phi \cup \mathrm{X}_{(\bar{\delta}|_i)}\big) \nvdash \bot$. A generating Δ-sequence is *closed* iff it is not a strict initial segment of any other generating Δ-sequence. Extensions and generating Δ-sequences are related as follows: $\mathrm{E} \in \mathsf{E}^{\Phi}_{\Delta}$ iff exists a generating Δ-sequence $\bar{\delta}$ s.t. $\mathrm{E} = \{\, \varepsilon \mid (\Phi \cup \mathrm{X}_{\bar{\delta}}) \vdash \varepsilon \,\}$. The proof of this fact can be obtained by adapting the one presented in [2].

Turning to the proof of Thm. 3.1, we first prove that if $\Phi \vdash \varphi$, then $\Phi \vdash^* \varphi$. Suppose that for a generating Δ-sequence $\bar{\delta}$, $\Phi \cup \mathrm{X}_{\bar{\delta}} \vdash \varphi$. From compactness for $\vdash$, we obtain that for some index n of $\bar{\delta}$, $\Phi \cup \mathrm{X}_{(\bar{\delta}|_n)} \vdash \varphi$. We convert a deduction of φ from $\Phi \cup \mathrm{X}_{(\bar{\delta}|_n)}$ into a default deduction of φ from Φ in the following way. For each $\chi_i \in \mathrm{X}_{(\bar{\delta}|_n)}$, there is $\delta_i = \pi_i/\chi_i \in \bar{\delta}|_n$; and so, there is a deduction $\bar{\psi}_{\pi_i}$ of π_i from $\Phi \cup \mathrm{X}_{(\bar{\delta}|_{(i-1)})}$. Construct a sequence $\bar{\psi}_{(\bar{\delta}|_n)} = \bar{\psi}_{\pi_1} \ldots \bar{\psi}_{\pi_n}$. Let $\bar{\psi}$ be a deduction of φ from $\Phi \cup \mathrm{X}_{(\bar{\delta}|_n)}$; the sequence $\bar{\psi}' = \bar{\psi}_{(\bar{\delta}|_n)}\bar{\psi}$ is a finite sequence of formulas which is, by construction, a default deduction of φ from Φ. It can easily be seen that $\bar{\psi}'$ is also credulous. Thus, if $\Phi \vdash \varphi$, then $\Phi \vdash^* \varphi$.

To prove that if $\Phi \vdash^* \varphi$, then $\Phi \vdash \varphi$, we assume that $\bar{\psi}$ is a credulous default deduction of φ from Φ. Let $\bar{\delta}$ be the Δ-sequence of defaults used in $\bar{\psi}$, i.e., those collected via default detachment; $\bar{\delta}$ is, by construction, a generating Δ-sequence. It can be proven that $\bar{\delta}$ can be extended to a generating Δ-sequence $\bar{\delta}'$ that is closed (see [3]). From this fact, the set $\mathrm{E} = \{\, \varepsilon \mid (\Phi \cup \mathrm{X}_{\bar{\delta}'}) \vdash_{\mathsf{M}^{\boxdot}} \varepsilon \,\}$ is an extension. Immediately, $\varphi \in \mathrm{E}$; and so $\Phi \vdash \varphi$. Thus, if $\Phi \vdash^* \varphi$, then $\Phi \vdash \varphi$.

In light of Thm. 3.1, we use $\vdash$ and $\vdash^*$ interchangeably.

3.3 Default Modal Systems Through an Algebraic Lens

We now turn our attention to viewing defaults and extensions in the setting of $\boxdot$-modal algebras. More precisely, we will focus on Lindenbaum-Tarski $\boxdot$-modal algebras. This view reveals how default modal systems may be thought of as systems with the ability of performing dynamic updates over a structure.

Remark 2. For the rest of this section, we assume $\Delta\mathsf{M}^{\boxdot} = \langle \mathsf{M}^{\boxdot}, \Delta, \mathsf{E} \rangle$ is an arbitary but fixed well-behaved default modal system. To simplify notation, we drop $\Delta\mathsf{M}^{\boxdot}$ and $\mathsf{M}^{\boxdot}$ as sub-scripts. Moreover, we write Φ, φ instead of $\Phi \cup \{\varphi\}$.

We construct this section around the following definition.

Definition 3.8. Let $\mathfrak{L} = \{\, \mathbf{L}^{\Phi} \mid \Phi \subseteq \mathsf{Form} \,\}$; for every default $\delta = \pi/\chi \in \Delta$; define a function $\hat{\delta} : \mathfrak{L} \to \mathfrak{L}$ s.t.:

$$\hat{\delta}(\mathbf{L}^{\Phi}) = \begin{cases} \mathbf{L}^{\Phi,\chi} & \text{if } [\pi]_{\Phi} = 1_{\Phi} \text{ and } 0_{\Phi} \notin \uparrow\{[\boxdot\chi]_{\Phi}\} \qquad (2a) \\ \mathbf{L}^{\Phi} & \text{otherwise.} \qquad (2b) \end{cases}$$

Def. 3.8 captures the effect of applying a default from an algebraic perspective. More precisely, applying a default $\delta = \pi/\chi$ w.r.t. a set Φ of formulas yields the set Φ, χ of formulas. The default is applicable iff: (a) $\Phi \vdash \pi$; and (b) $\Phi, \chi \nvdash \bot$. In algebraic terms, we capture the application of a default as a transformation between LT $\boxdot$-modal algebras. More precisely, consider the LT $\boxdot$-modal algebra for a set Φ of formulas $\mathbf{L}^{\Phi}$. The condition (a) of applicability of $\delta = \pi/\chi$ w.r.t. $\mathbf{L}^{\Phi}$ is captured in (2a) as $[\pi]_{\Phi} = 1_{\Phi}$; and the condition (b) of applicability is captured in (2a) as $0_{\Phi} \notin \uparrow\{[\boxdot\chi]_{\Phi}\}$. In other words, the equivalence class of 1_{Φ} captures the deducibility of π from Φ. In turn, the condition of *being proper* on the (open) filter generated by $[\boxdot\chi]_{\Phi}$ captures the consistency of χ w.r.t. Φ. Notice that if the default is applicable, the return value of $\hat{\delta}$, i.e., $\mathbf{L}^{\Phi,\chi}$, is tantamount to incorporating χ to Φ. Contrariwise, i.e., if δ is not applicable, $\hat{\delta}$ has no effect on $\mathbf{L}^{\Phi}$. When seen in this light, the operator $\hat{\delta}$ performs an *update* reflecting the application of δ on its input. The situation here is similar to the case in logics of updates such as Public Announcement Logic [24] (in particular, in relation to the approach proposed in [23]). We retake this discussion in Sec. 4.

Having looked at the effect of defaults from an algebraic perspective, we turn our attention to constructing extensions. For well-behaved defaults, extensions can be seen as being constructed in a step-wise fashion applying defaults one at a time. From a syntactic perspective, this construction of an extension starts with a closed set Φ, and applies the defaults $\delta \in \Delta$ one by one until we obtain a closed set of formulas that is saturated under the application of defaults. From the perspective of LT $\boxdot$-modal algebras we obtain the following.

Proposition 3.3. Each function $\hat{\delta}$ induces a function $\bar{\delta} : |\mathbf{L}| \to |\hat{\delta}(\mathbf{L})|$ defined as: $\bar{\delta}([\varphi]_{\Phi}) = [\varphi]_{\Phi,\chi}$ if Eq. (2a) holds; or $\bar{\delta}([\varphi]_{\Phi}) = [\varphi]_{\Phi}$ if Eq. (2b) holds. The function $\bar{\delta}$ is a homomorphism from $\mathbf{L}$ to $\hat{\delta}(\mathbf{L})$.

Proof. That $\bar{\delta}$ is a function is trivial. The proof that $\bar{\delta}$ is a homomorphism is by cases. If Eq. (2b) holds, then, the result is obtained immediately. Otherwise:

$$\bar{\delta}(f^{\Box}_{\Phi}([\varphi]_{\Phi})) = \bar{\delta}([\Box\varphi]_{\Phi}) = [\Box\varphi]_{\Phi,\chi} = f^{\Box}_{\Phi,\chi}([\varphi]_{\Phi,\chi}) = f^{\Box}_{\Phi,\chi}(\bar{\delta}([\varphi]_{\Phi})).$$

The remaining cases are similar.

The following are some immediate properties of default operators.

Definition 3.9. Let $\mathbf{L}_1, \mathbf{L}_2 \in \mathfrak{L}$; we write $\mathbf{L}_1 \leq \mathbf{L}_2$ iff there is a homomorphism $h : \mathbf{L}_1 \to \mathbf{L}_2$; and $\mathbf{L}_1 < \mathbf{L}_2$ iff $\mathbf{L}_1 \leq \mathbf{L}_2$ and $\mathbf{L}_1$, $\mathbf{L}_2$ are not isomorphic.

Proposition 3.4. Every $\hat{\delta}$ is extensive and idempotent, i.e., it satisfies $\mathbf{L} \leq \hat{\delta}(\mathbf{L})$ and $\hat{\delta}(\mathbf{L}) = \hat{\delta}(\hat{\delta}(\mathbf{L}))$, respectively. An arbitrary $\hat{\delta}$ needs not satisfy monotonicity, i.e., there are $\delta = \pi/\chi$ s.t. $\mathbf{L}_1 \leq \mathbf{L}_2$ and $\hat{\delta}(\mathbf{L}_1) \not\leq \hat{\delta}(\mathbf{L}_2)$.

Proof. Extensivity follows from Prop. 3.3. Idempotence is proven by cases. If Eq. (2b) holds, then, the result is obtained immediately. Otherwise, Eq. (2a) holds. In this case, $\hat{\delta}(\mathbf{L}^{\Phi}) = \mathbf{L}^{\Phi,\chi}$. Trivially, $\hat{\delta}(\mathbf{L}^{\Phi,\chi}) = \mathbf{L}^{\Phi,\chi}$. For a counter-example to monotonicity, consider LT $\boxdot$-modal algebras $\mathbf{L}^{\emptyset}_{\mathsf{K}^{\boxdot}}$ and $\mathbf{L}^{\{\Box p\}}_{\mathsf{K}^{\boxdot}}$ and a default $\delta = \top/\Diamond\neg p$. Obviously, $\mathbf{L}^{\emptyset}_{\mathsf{K}^{\boxdot}} \leq \mathbf{L}^{\{\Box p\}}_{\mathsf{K}^{\boxdot}}$. However, there is no homomorphism from $\hat{\delta}(\mathbf{L}^{\emptyset}_{\mathsf{K}^{\boxdot}})$ to $\hat{\delta}(\mathbf{L}^{\{\Box p\}}_{\mathsf{K}^{\boxdot}})$.

Any set Δ of well-behaved defaults leads naturally to a set $\{\, \hat{\delta} \mid \delta \in \Delta \,\}$. Each $\hat{\delta}$ in this set can be seen as "taking a step" in the construction of the algebraic counterpart of an extension. To carry out this construction in its entirety, we would need to compose such steps. This leads to the formulation of Def. 3.10.

Definition 3.10. Let $\mathbf{D}$ be the monoid freely generated by $\{\, \hat{\delta} \mid \delta \in \Delta \,\}$, i.e., $\mathbf{D} = \langle \mathrm{D}, -;-, \mathrm{id} \rangle$ where:

1. D is the $\subseteq$-smallest set s.t.:
   (a) $\{\, \hat{\delta} : \mathfrak{L} \to \mathfrak{L} \mid \delta \in \Delta \,\} \subseteq \mathrm{D}$;
   (b) $\mathrm{id} : \mathfrak{L} \to \mathfrak{L} \in \mathrm{D}$; and
   (c) if $\{d_1 : \mathfrak{L} \to \mathfrak{L}, d_2 : \mathfrak{L} \to \mathfrak{L}\} \subseteq \mathrm{D}$, then $(d_1;d_2) : \mathfrak{L} \to \mathfrak{L} \in \mathrm{D}$;
2. id and $-;-$ satisfy: $\mathrm{id}(\mathbf{L}) = \mathbf{L}$; and $(d_1;d_2)(\mathbf{L}) = d_2(d_1(\mathbf{L}))$.

We refer to $\mathbf{D}$ as the *default monoid* (associated to the default modal system).

Proposition 3.5. Every $d \in |\mathbf{D}|$ is either: the identity, i.e., $d = \mathrm{id}$; or a composition of the form $d = (\hat{\delta}_1; \ldots ;\hat{\delta}_n)$, where $\delta_i \in \Delta$.

Definition 3.11. Let $\mathbf{D}$ be a default monoid, $\mathbf{L}$ be a LT $\boxdot$-modal algebra, and v be an assignment on $\mathbf{L}$; for every equation $\varphi \approx \psi$, define:

$$\mathbf{L}, v \approx\!\!\!\!\models \varphi \approx \psi \quad \text{iff} \quad d(\mathbf{L}), (v;\bar{d}) \models \varphi \approx \psi \text{ for some } d \in |\mathbf{D}|.$$

where $\overline{\mathrm{id}}([\varphi]_{\Phi}) = [\varphi]_{\Phi}$; and $\overline{(\hat{\delta}_1; \ldots ;\hat{\delta}_n)} = (\bar{\delta}_1; \ldots ;\bar{\delta}_n)$. We write $\mathbf{L} \approx\!\!\!\!\models \varphi \approx \psi$ iff $\mathbf{L}, v \approx\!\!\!\!\models \varphi \approx \psi$ for all assignments v on $\mathbf{L}$. Moreover, we write $\approx\!\!\!\!\models \varphi \approx \psi$ iff $\mathbf{L} \approx\!\!\!\!\models \varphi \approx \psi$ for all LT $\boxdot$-modal algebras $\mathbf{L}$.

Intuitively, the LT $\boxdot$-modal algebra $d(\mathbf{L})$ in Def. 3.11 is the algebraic version of the concept of an extension. This is made clear in Thm. 3.2.

Theorem 3.2. For all sets of formulas Φ, φ, we have $\Phi \vdash \varphi$ iff $\mathbf{L}^{\Phi} \approx\!\!\!\!\models \varphi \approx \top$.

Proof. The interesting part is the right-to-left implication: if $\mathbf{L}^{\Phi} \approx\!\!\!\!\models \varphi \approx \top$, then, $\Phi \vdash \varphi$. We prove the contrapositive: if $\Phi \not\vdash \varphi$, then, $\mathbf{L}^{\Phi} \not\approx\!\!\!\!\models \varphi \approx \top$.

Let $\Phi \not\vdash \varphi$, the proof is concluded if for all $d \in |\mathbf{D}|$, $d(\mathbf{L}^\Phi) \not\models \varphi \approx \top$. We continue by induction on d.

Base case: let $d = \mathrm{id}$; we must have $\mathrm{id}(\mathbf{L}^\Phi) \not\models \varphi \approx \top$; otherwise we would obtain $\Phi \vdash \varphi$ (from Thm. 2.1); and so that $\Phi \mathrel{|\!\sim} \varphi$ (which contradicts our assumption).

Base case: let $d = \hat{\delta}$ for $\delta = \pi/\chi \in \Delta$; either Eq. (2b) holds or Eq. (2a) holds. If Eq. (2b) holds, $\hat{\delta}$ behaves like id (and we are back to the previous case). If Eq. (2a) holds, $\hat{\delta}(\mathbf{L}^\Phi) = \mathbf{L}^{\Phi,\chi}$. Assuming (i) $\mathbf{L}^{\Phi,\chi} \models \varphi \approx \top$ leads to a contradiction. More precisely, if Eq. (2a) holds, from Thm. 2.1, we obtain $\Phi \vdash \pi$ and $\Phi, \chi \not\vdash \bot$. From (i) and Thm. 2.1, we obtain $\Phi, \chi \vdash \varphi$. If we place the deduction of π from Φ in front of the deduction of φ from Φ, χ, we obtain a default deduction of φ from Φ. This yields a contradiction.

Inductive case: let $d = (\hat{\delta}_1; \ldots; \hat{\delta}_n; \hat{\delta}_{(n+1)})$. If $(\hat{\delta}_1; \ldots; \hat{\delta}_n)(\mathbf{L}^\Phi) = \mathbf{L}^{\Phi'}$, from the inductive hypothesis, we obtain $\mathbf{L}^{\Phi'} \not\models \varphi \approx \top$. Assuming that $\hat{\delta}_{(n+1)}(\mathbf{L}^{\Phi'}) \models \varphi \approx \psi$ leads to a contradiction using the same argument as in (i).

We conclude this section by taking some steps beyond dealing with defaults and extensions in the context of LT $\boxbox$-modal algebras. In particular, we show how some of the constructions used in Sec. 3.3 can be extended to a more abstract setting via suitable congruences.

Definition 3.12. Let $\mathbf{L}^\Phi$ be a Lindenbaum-Tarski $\boxbox$-modal algebra and χ a formula; define $[\varphi_1]_\Phi \equiv_\chi [\varphi_2]_\Phi$ iff $[\varphi_1]_\Phi *_\Phi [\boxbox\chi]_\Phi = [\varphi_2]_\Phi *_\Phi [\boxbox\chi]_\Phi$.

Def. 3.12 is a step towards treating the application of default as a device for obtaining a $\boxbox$-modal algebra $\mathbf{M}$ updated by the element $[\chi]_\Phi$ in $\mathbf{L}^\Phi$. The updated $\boxbox$-modal algebra $\mathbf{M}$ is meant to be obtained as a quotient algebra modulo the congruence $\equiv_\chi$. Prop. 3.6 shows that $\equiv_\chi$ indeed is a congruence.

Proposition 3.6. The relation $\equiv_\chi$ is a congruence on $\mathbf{L}^\Phi$.

Proof. That $\equiv_\chi$ is an equivalence relation is immediate. To improve notation we drop the subscript $_\Phi$. We need to show that: if $[\varphi_1] \equiv_\chi [\varphi_2]$ and $[\varphi_3] \equiv_\chi [\varphi_4]$, then, $[\varphi_1] * [\varphi_3] \equiv_\chi [\varphi_2] * [\varphi_4]$; $-[\varphi_1] \equiv_\chi -[\varphi_2]$; $f^\Box([\varphi_1]) \equiv_\chi f^\Box([\varphi_2])$; and $f^\boxbox([\varphi_1]) \equiv_\chi f^\boxbox([\varphi_2])$. The proof continues by cases (we only show the cases $f^\Box$ and $f^\boxbox$, the rest are routine):

$$\begin{aligned}
&f^\Box([\varphi_1]) * [\boxbox\chi] \\
&\quad \geq f^\Box([\varphi_1] * [\boxbox\chi]) * [\boxbox\chi] \\
&\quad = f^\Box([\varphi_2] * [\boxbox\chi]) * [\boxbox\chi] \\
&\quad = f^\Box([\varphi_2]) * (f^\Box([\boxbox\chi]) * [\boxbox\chi]) \\
&\quad \geq f^\Box([\varphi_2]) * [\boxbox\chi]
\end{aligned}$$

$$\begin{aligned}
&f^\boxbox([\varphi_1]) * [\boxbox\chi] \\
&\quad = f^\boxbox([\varphi_1]) * [\boxbox\boxbox\chi] \\
&\quad = f^\boxbox([\varphi_1]) * f^\boxbox([\boxbox\chi]) \\
&\quad = f^\boxbox([\varphi_1] * [\boxbox\chi]) \\
&\quad = f^\boxbox([\varphi_2] * [\boxbox\chi]) \\
&\quad = f^\boxbox([\varphi_2]) * f^\boxbox([\boxbox\chi]) \\
&\quad = f^\boxbox([\varphi_2]) * [\boxbox\boxbox\chi] \\
&\quad = f^\boxbox([\varphi_2]) * [\boxbox\chi].
\end{aligned}$$

Proposition 3.7. The quotient algebra $\mathbf{L}^\Phi/_{\equiv_\chi}$ is isomorphic to $\mathbf{L}^{\Phi,\chi}$.

Proof. Observe that $\Phi, \chi \vdash (\varphi_1 \leftrightarrow \varphi_2)$ iff $\Phi \vdash (\varphi_1 \wedge \boxbox\chi \leftrightarrow \varphi_2 \wedge \boxbox\chi)$. The isomorphism between $\mathbf{L}^\Phi/_{\equiv_\chi}$ and $\mathbf{L}^{\Phi,\chi}$ is given by mappings ι_1 and ι_2 defined as: $\iota_1([[\varphi]_\Phi]_{\equiv_\chi}) = [\varphi]_{\Phi,\chi}$; and $\iota_2([\varphi]_{\Phi,\chi}) = [[\varphi]_\Phi]_{\equiv_\chi}$.

The isomorphism in Prop. 3.7 shows that the relation $\equiv_\chi$ yields the "correct" congruence if the application of a default is to be seen as updating a $\boxdot$-modal algebra. Moreover, it is possible to define a function $\varepsilon : \mathbf{L}^\Phi/_{\equiv_\chi} \to \mathbf{L}^\Phi$ defined by $\varepsilon([[\varphi]_\Phi]_{\equiv_\chi}) = [\varphi]_\Phi *_\Phi [\chi]_\Phi$. The image of ε is also isomorphic to $\mathbf{L}^{\Phi,\chi}$. The results discussed in this paragraph open a pathway on how to lift the constructions in Defs. 3.8 and 3.10 to the setting of arbitrary $\boxdot$-modal algebras and to connect default modal systmes with logics of updates.

3.4 An Application of the Algebraic Framework

The results in Sec. 3.3 shows us how to deal with default modal systems using algebraic tools. Interestingly, this enables us to formulate and obtain a completeness result using LT $\boxdot$-modal algebras. The algebraic machinery in Sec. 3.3 also opens a pathway to investigate other properties of default modal systems. In particular, in this section, we show how the notion of bisimulation for modal logics can be adapted and extended to enable us to compare default modal systems.

We begin by recalling the standard definition of Kripke models, and the semantics of modal formulas.

Definition 3.13. A *frame* is a tuple $\mathfrak{F} = \langle W, R\rangle$ where: W is a set of elements (called *worlds*); and $R \subseteq W^2$ is the *accessibility relation*. A *Kripke model* is a pair $\mathfrak{M} = \langle \mathfrak{F}, V\rangle$ where: $\mathfrak{F}$ is a frame, and $V : \mathsf{Prop} \to 2^W$ is the *valuation function*. For $w \in W$, the pair $\mathfrak{M}, w$ is called a *pointed model*.

Definition 3.14. Let $\mathfrak{M} = \langle W, R, V\rangle$ be a Kripke model, $w \in W$, and $\varphi \in \mathsf{Form}$; the *satisfiability relation* $\mathfrak{M}, w \Vdash \varphi$ is defined according to the following rules:

$$\begin{array}{lll}
\mathfrak{M}, w \Vdash p_i & \text{iff} & w \in V(p_i) \\
\mathfrak{M}, w \Vdash \neg\varphi & \text{iff} & \mathfrak{M}, w \nVdash \varphi \\
\mathfrak{M}, w \Vdash \varphi \vee \psi & \text{iff} & \mathfrak{M}, w \Vdash \varphi \text{ or } \mathfrak{M}, w \Vdash \psi \\
\mathfrak{M}, w \Vdash \Box\varphi & \text{iff} & \text{for all } w' \in W,\ wRw' \text{ implies } \ \mathfrak{M}, w' \Vdash \varphi \\
\mathfrak{M}, w \Vdash \boxdot\varphi & \text{iff} & \text{for all } w' \in W,\ \mathfrak{M}, w' \Vdash \varphi.
\end{array}$$

A Kripke model $\mathfrak{M} = \langle W, R, V\rangle$ *satisfies* a formula φ at a world $w \in W$ iff $\mathfrak{M}, w \Vdash \varphi$; and it *validates* φ, written $\mathfrak{M} \Vdash \varphi$, iff $\mathfrak{M}, w \Vdash \varphi$, for all $w \in W$. The model $\mathfrak{M}$ *satisfies* a set of formulas Φ at w, notation $\mathfrak{M}, w \Vdash \Phi$, if $\mathfrak{M}, w \Vdash \varphi$ for all $\varphi \in \Phi$. And it *validates* Φ, notation $\mathfrak{M} \Vdash \Phi$, if $\mathfrak{M}, w \Vdash \Phi$ for all $w \in W$.

The following proposition links Kripke models and the algebraic structures introduced in Sec. 2.3 (the full details can be found, e.g., in [32]).

Definition 3.15. Let $\mathbf{M} = \langle B, *, -, 1, f^\Box, f^\boxdot\rangle$ be a $\boxdot$-modal algebra; its dual, written $\mathbf{M}^\bullet$, is a frame $\langle \mathrm{Uf}(\mathbf{M}), R\rangle$ where: $\mathrm{Uf}(\mathbf{M})$ is the set of all ultrafilters in $\mathbf{M}$ and R is defined by $u_1 \mathrel{R} u_2$ iff $-f^\Box(-a) \in u_1$ for all $a \in u_2$. The dual of an interpretation $v : \mathbf{F} \to \mathbf{M}$ is a function $v^\bullet : \mathsf{Prop} \to 2^{\mathrm{Uf}(\mathbf{M})}$ defined as: $v^\bullet(p) = \{\, u \in \mathrm{Uf}(\mathbf{M}) \mid v(p) \in u \,\}$. We define $(\mathbf{M}, v)^\bullet = (\mathbf{M}^\bullet, v^\bullet)$.

Proposition 3.8. Let $\mathbf{M}$ be a $\boxdot$-modal algebra and $v : \mathbf{F} \to \mathbf{M}_i$ be an interpretation on $\mathbf{M}$; for all equations $\varphi \approx \psi$:

$$\mathbf{M}, v \vDash \varphi \approx \psi \text{ iff } (\mathbf{M}, v)^\bullet \Vdash \varphi \leftrightarrow \psi.$$

Let us recall the standard notion of bisimulation for Kripke models [6].

Definition 3.16. For $i \in \{1,2\}$, let $\mathfrak{M}_i = \langle W_i, R_i \subseteq W_i^2, V_i : \mathsf{Prop} \to 2^{W_i} \rangle$ be a Kripke model; a (non-empty) relation $Z \subseteq W_1 \times W_2$ is a *bisimulation* between $\mathfrak{M}_1$ and $\mathfrak{M}_2$ iff $w_1 \, Z \, w_2$ implies

(atom) $w_1 \in V_1(p)$ iff $w_2 \in V_2(p)$, for all $p \in \mathsf{Prop}$;

(forth) if $w_1 \, R_1 \, w_3$, there is $w_4 \in W_2$ s.t. $w_2 \, R_2 \, w_4$ and $w_3 \, Z \, w_4$;

(back) if $w_2 \, R_2 \, w_4$, there is $w_3 \in W_1$ s.t. $w_1 \, R_1 \, w_3$ and $w_3 \, Z \, w_4$.

The bisimulation Z is total iff: for all $w_1 \in W_1$, exists $w_2 \in W_2$ s.t. $w_1 \, Z \, w_2$; and for all $w_2 \in W_2$, exists $w_1 \in W_1$ s.t. $w_1 \, Z \, w_2$. We write $\mathfrak{M}_1 \leftrightarroweq \mathfrak{M}_2$ iff there is a total bisimulation Z between $\mathfrak{M}_1$ and $\mathfrak{M}_2$; and $\mathfrak{M}_1, w_1 \leftrightarroweq \mathfrak{M}_2, w_2$ iff $w_1 \, Z \, w_2$.

Prop. 3.9 states a well-known result regarding bisimulations.

Proposition 3.9. If $\mathfrak{M}_1 \leftrightarroweq \mathfrak{M}_2$, then, $\mathfrak{M}_1 \Vdash \varphi$ iff $\mathfrak{M}_2 \Vdash \varphi$.

Prop. 3.9 has an analogous proposition in terms of $\boxdot$-modal algebras.

Definition 3.17. Let $\mathbf{M}_1$ and $\mathbf{M}_2$ be two $\boxdot$-modal algebras and v_1 and v_2 be interpretations on $\mathbf{M}_1$ and $\mathbf{M}_2$, respectively; we write $(\mathbf{M}_1, v_1) \leftrightarroweq (\mathbf{M}_2, v_2)$ iff $(\mathbf{M}_1, v_1)^{\bullet} \leftrightarroweq (\mathbf{M}_2, v_2)^{\bullet}$.

Proposition 3.10. Let $\mathbf{M}_1$ and $\mathbf{M}_2$ be two $\boxdot$-modal algebras and v_1 and v_2 be interpretations on $\mathbf{M}_1$ and $\mathbf{M}_2$, respectively; if $(\mathbf{M}_1, v_1) \leftrightarroweq (\mathbf{M}_2, v_2)$, then, $\mathbf{M}_1, v_1 \vDash \varphi \approx \psi$ iff $\mathbf{M}_2, v_2 \vDash \varphi \approx \psi$.

We are now in a position to show how to extend and adapt the concept of bisimulation to the algebraic treatment of defaults and extensions.

Definition 3.18. Let $\mathbf{D}_1$ and $\mathbf{D}_2$ be the default monoids associated to two default modal systems built on the same underlying modal system; a (non-empty) relation $Z \subseteq \mathfrak{L} \times \mathfrak{L}$ is a *default bisimulation* iff $\mathbf{L}_1 \, Z \, \mathbf{L}_2$ implies:

(atom) $(\mathbf{L}_1, v_1) \leftrightarroweq (\mathbf{L}_2, v_2)$ where $v_i : \mathbf{F} \to \mathbf{L}_i$ is the canonical interpretation;

(forth) for all $d_1 \in |\mathbf{D}_1|$, there is $d_2 \in |\mathbf{D}_2|$ s.t. $d_1(\mathbf{L}_1) \, Z \, d_2(\mathbf{L}_2)$;

(back) for all $d_2 \in |\mathbf{D}_2|$, there is $d_1 \in |\mathbf{D}_1|$ s.t. $d_1(\mathbf{L}_1) \, Z \, d_2(\mathbf{L}_2)$.

We write $\mathbf{D}_1 \leftrightarroweq \mathbf{D}_2$ iff there is a default bisimulation Z between $\mathbf{D}_1$ and $\mathbf{D}_2$; and $\mathbf{D}_1, \mathbf{L}_1 \leftrightarroweq \mathbf{D}_2, \mathbf{L}_2$ iff $\mathbf{L}_1 \, Z \, \mathbf{L}_2$.

The definition of a default bisimulation in Def. 3.18 enables us to compare when two default monoids are indistinguisable; or, what is the same, it enables us to compare when two default modal systems are indistinguisable. This fact is stated in Prop. 3.11.

Proposition 3.11. If $\mathbf{D}_1, \mathbf{L}_1 \leftrightarroweq \mathbf{D}_2, \mathbf{L}_2$, then $\mathbf{L}_1 \approx\!\!\!\vDash_1 \varphi \approx \psi$ iff $\mathbf{L}_2 \approx\!\!\!\vDash_2 \varphi \approx \psi$; where $\approx\!\!\!\vDash_i$ is the relation in Def. 3.11 formulated w.r.t. $\mathbf{D}_i$.

Proof. Let Z be a default bisimulation between $\mathbf{D}_1$ and $\mathbf{D}_2$, and $\mathbf{L}_1 \, Z \, \mathbf{L}_2$. We prove the left-to-right direction, i.e., if $\mathbf{L}_1 \approx\!\!\!\vDash_1 \varphi \approx \psi$, then, $\mathbf{L}_2 \approx\!\!\!\vDash_2 \varphi \approx \psi$. Assume $\mathbf{L}_1 \approx\!\!\!\vDash_1 \varphi \approx \psi$. There is $d_1 \in |\mathbf{D}_1|$ s.t. $d_1(\mathbf{L}_1) \vDash \varphi \approx \psi$. In particular, $d_1(\mathbf{L}_1), v_1 \vDash \varphi \approx \psi$ for $v_1 : \mathbf{F} \to f(\mathbf{L}_1)$ the canonical interpretation on $\mathbf{L}_1$. From (forth), there is $d_2 \in |\mathbf{D}_2|$ s.t. $d_1(\mathbf{L}_1) \, Z \, d_2(\mathbf{L}_2)$. Then, from (atom), we obtain $(d_1(\mathbf{L}_1), v_1) \leftrightarroweq (d_2(\mathbf{L}_2), v_2)$ for $v_2 : \mathbf{F} \to f_2(\mathbf{L}_2)$ the canonical interpretation on $\mathbf{L}_2$. Using Prop. 3.10, it follows that $d_2(\mathbf{L}_2), v_2 \vDash \varphi \approx \psi$. Since canonical interpretations are closed under substitution, $f_2(\mathbf{L}_2) \vDash \varphi \approx \psi$. Therefore, $\mathbf{L}_2 \approxeq_2 \varphi \approx \psi$. The right-to-left direction follows the same argument using (back) instead of (forth).

We conclude this section with a comment on an application of the concept of bisimulation between default monoids to the problem of equivalence of so-called *default theories*. More precisely, in Default Logic, a default theory is a pair $\langle \Phi, \Delta \rangle$ where Φ is a set of formulas and Δ is a set of defaults. The problem of equivalence of default theories pertains to the following question: in which sense two default theories $\langle \Phi_1, \Delta_1 \rangle$ and $\langle \Phi_2, \Delta_2 \rangle$ can be regarded as equivalent? Adequate answers to this question have a bearing in the setting of transformation of logical programs, e.g., to improve efficiency, as default theories may be seen as the logical counterpart of logical programs. It is common, e.g., [27, 20], to address equivalence of default theories from a syntactic point of view by focusing on what is the case w.r.t. extensions. Under such a point of view, a default theory $\langle \Phi_1, \Delta_1 \rangle$ is deemed equivalent to a default theory $\langle \Phi_2, \Delta_2 \rangle$ whenever $\mathsf{E}^{\Phi_1}_{\Delta_1} = \mathsf{E}^{\Phi_2}_{\Delta_2}$; and they are deemed to be strongly equivalent whenever $\langle \Phi_1 \cup \Phi_3, \Delta_1 \cup \Delta_3 \rangle$ and $\langle \Phi_2 \cup \Phi_3, \Delta_2 \cup \Delta_3 \rangle$ are equivalent for all default theories $\langle \Phi_3, \Delta_3 \rangle$. By way of example, $\langle \{\Box p\}, \emptyset \rangle$ and $\langle \emptyset, \{\top / \Box p\} \rangle$ are equivalent, but they are not strongly equivalent (as $\langle \{\Diamond \neg p\}, \emptyset \rangle$ distinguishes them). The concept of bisimulation between default monoids enables us to look at the problem of equivalence of default theories in a new light, i.e., from a semantic perspective. For a fixed underlying modal system, we could say that $\langle \Phi_1, \Delta_1 \rangle$ and $\langle \Phi_2, \Delta_2 \rangle$ are equivalent whenever $\mathbf{D}_1, \mathbf{L}^{\Phi_1} \leftrightarroweq \mathbf{D}_2, \mathbf{L}^{\Phi_2}$ (where $\mathbf{D}_1$ and $\mathbf{D}_2$ are the default monoids associated to the default modal systems constructed relative to Δ_1 and Δ_2, resp). The notion of strong equivalence is defined in a similar way. To be noted, our definition of equivalence is given in semantic terms building on the notion of bisimulation for the underlying modal system. To be noted also, our definition of equivalence pays closer attention to the different elements of a default theory (i.e., its set of formulas and its set of defaults). For instance, notice that under our definition, $\langle \{\Box p\}, \emptyset \rangle$ and $\langle \emptyset, \{\top / \Box p\} \rangle$ are not equivalent, as $\mathbf{L}^{\{\Box p\}}, v_1 \not\leftrightarroweq \mathbf{L}^{\emptyset}, v_2$ for v_1 and v_2 the canonical interpretations on the corresponding LT $\boxdot$-modal algebras. Looking at problems of equivalence from a semantic perspective has a long standing tratition in the field of Modal Logic. In this respect, bisimulations have proven to be a useful tool. The concept of bisimulation between default monoids is a step in this direction. It is worth noting that a practical advantage of bisimulations over other techniques for proving semantic equivalence is given by checking properties orelative to given points. In our case, this boils down to prove properties of the Lindenbaum-Tarski algebras. This has a "local" flavor that is in contrast to proving equivalence between default theories which are, in some sense, "global" in nature; in that they need to look into the entire collection of extensions. Furthermore, it is worth noting that bisimulations can be algebraized via coalgebras. In this respect, our work seems a natural step towards the definition an algebraic toolset for reasoning about default modal systems in a mathematical setting. As further work also, it remains to generalize the concept of bisimulation between default monoids along two main lines: default operators defined relative to arbitrary defaults, and arbitrary $\boxdot$-modal algebras. This generalization seems an ideal tool for discussing the deductive aspects of default theories independently of their particular syntactical constructs.

4 On Defaults as Model Updates

We are now in a position to establish a connection between our algebraic approach for default modal systems and the algebraic treatment of Public Announcement Logic (PAL) in [23].

To set up context for discussion, we briefly introduce some basic notions of PAL (see, e.g., [24] for details). As a modal logic, PAL extends the modal logic S5 (seen as the logic of knowledge) with a new modality $\langle !\psi\rangle$ of announcement, defined as:

$$\mathfrak{M}, w \Vdash \langle !\psi\rangle\varphi \quad \text{iff} \quad \mathfrak{M}, w \Vdash \psi \text{ implies } \mathfrak{M}_{|\psi}, w \Vdash \varphi, \tag{3}$$

where $\mathfrak{M}_{|\psi}$ is the restriction of $\mathfrak{M}$ to those states in which ψ hods. Intuitively, a formula $\langle !\psi\rangle\varphi$ states that after the truthful announcement of ψ, φ holds. Model theoretically, the interpretation of announcing ψ relativizes the model in which ψ is announced to the submodel in which ψ holds everywhere. The formula φ is then evaluated on the relativized model. It should be noted that the announcement of ψ must be truthful: it occurs only if ψ is true. Otherwise, the announcement "fails" and $\langle !\psi\rangle\varphi$ evaluates to false.

There are some interesting similarities between announcements in PAL and defaults. From an algebraic perspective, an announcement may be understood as a homomorphism between the modal algebra in which the announcement occurs and the modal algebra corresponding to the submodel in which the announced formula holds. This is the approach taken in [23]. In the default case, the algebraic machinery introduced in Sec. 3.3 sets the basis for thinking about the application of defaults as a logic of updates between particular modal algebras (LT $\boxminus$-modal algebras). More precisely, we may construe the algebraic semantics of a default as an update from the LT $\boxminus$-modal algebra in which the default is considered, to the LT $\boxminus$-modal algebra updated with the consequent of the default (if the default is applicable). Notice that a default update takes place only if the prerequisite of the default is provable and its consequent does not yield an inconsistency. The situation here is similar to the case of announcements, where the update takes place only if the formula being announced is true (see Eq. (3)). In both cases, that of an announcement and that of the algebraic application of a default, the update is captured by a homomorphism from the original modal algebra to an updated modal algebra (obtained as a quotient construction). There is, however, a subtle difference between announcements and defaults: if the announcement of ψ is not truthful the whole formula $\langle !\psi\rangle\varphi$ amounts to a falsity; whereas if the prerequisite of a default is not provable, or its consequent is inconsistent in the modal algebra, the application of the default has no effect.

The similarities between announcements in PAL and defaults are even more apparent when contrasted with the way in which announcements are dealt with in [23]. In particular, the approach in [23] exploits the duality between models and algebras in order to algebraize PAL. Therein, a formula ψ is interpreted as an element b in an S5 modal algebra $\mathbf{M} = \langle B, *, -, f^{\Box}\rangle$. The result of announcing this formula is a modal algebra constructed as a quotient modulo a congruence $\equiv_b$ defined as:

$$b_1 \equiv_b b_2 \quad \text{iff} \quad b_1 * b = b_2 * b.$$

This congruence bears a close resemblance to the one we presented in Sec. 3.3.

The main difference between this congruence and ours rests on the fact that the former is presented in the setting of $\mathsf{S5}$, whereas ours is presented in a setting where global modal consequence is taken as the basis on which to build default modal systems. This said, the approach in [23] is more abstract than ours; since it considers arbitrary modal algebras and not just those obtained via Lindenbaum-Tarski constructions.

The discussion above offers only some first steps in understanding the relationship between defaults and updates: both in terms of a full algebraization of default modal systems, and in terms of establishing a tight connection with logics of updates. In working towards a full algebraization of default modal systems, we would like to interpret the application of a default over arbitrary modal algebras, and not only as an update over LT $\boxed{\mathsf{u}}$-modal algebras. In this regard, the main challenge is how to generalize the way in which we capture the application of one default to the application of a sequence of defaults needed to build an extension. Moreover, it would also be interesting to know whether it is possible to develop a class of algebraic structures for default modal systems parallel to the class of modal algebras for modal systems. This would require an internalization of defaults as algebraic operators. In turn, in what refers to establishing a tight connection with logics of updates, it would be interesting to be able to prove a reduction result between a default modal system and a logic of announcements (or establishing a difference in expressive power between one and the other). In this case, the challenge is deciding on an adequate logic of announcements and in finding whether it is possible to faithfully translate the application of a default as a form of update in this logic. Finally, upon defining the semantics of defaults as updates, we would like to study defaults as dynamic epistemic operators. In particular, we would like to explore whether defaults can be used to represent some novel form of communication between agents in a multi-agent setting.

5 Final Remarks

We presented a family of default logics built on modal logics ranging from K to $\mathsf{S5}$, and studied some of their properties. We approached this presentation, first, syntactically via what we called a default modal system. For each default modal system we formulated a notion of default deducibility to make explicit how defaults interact with deducibility in the underlying modal system. Then, we offered an algebraic treatment of defaults and extensions, via transformations on LT $\boxed{\mathsf{u}}$-modal algebras and default monoids, respectively. The algebraic treatment enabled us to obtain an algebraic completeness result. Interestingly enough, this approach also enabled us to think of a way of comparing default logics by borrowing ideas from the concept of a bisimulation in modal logic. To our knowledge, this is the first work addressing default logic algebraically.

There are several interesting lines for future work. We do notice that our work is not an algebraization of a logic. Instead, we have taken advantage of algebraic tools to study default modal systems from a semantic perspective. In this respect, our work is a first step towards an algebraization of default modal systems. There is still a need to identify a class of algebras for default modal systems that would play a role akin to $\boxed{\mathsf{u}}$-modal algebras in modal systems. We also wish to generalize our constructions and results to arbitrary $\boxed{\mathsf{u}}$-modal algebras and not just LT ⊡-modal algebras. We consider that the trail of coalgebras (see [33]) may provide an adequate abstract framework in which to generalize and further investigate our ideas. Bisimulations can be algebraized via coalgebras. In this respect, our work on bisimulations between default monoids is a natural step towards the definition an algebraic toolset for reasoning about default modal systems in a mathematical setting. Furthermore, we wish to exploit our algebraic treatment of default modal systems to study semantic properties such as: invariance and Hennessy-Milner theorems, interpolation, Beth definability, etc.

Lastly, our characterization of defaults as transformations on ⊡-modal algebras works as a sort of "update". It would interesting to find connections with algebraic approaches to logics with update operators, in the sense of e.g. Public Announcement Logic. The work reported in [23] seems to shed some light in this direction.

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