Interpolation and Beth Definability in Default Logics

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Abstract. We investigate interpolation and Beth definability in default logics. To this end, we start by defining a general framework which is sufficiently abstract to encompass most of the usual definitions of a default Logic. In this framework a default logic $\mathscr{D}\mathfrak{L}$ is built on a base, monotonic, logic $\mathfrak{L}$. We then investigate the question of when interpolation and Beth definability results transfer from $\mathfrak{L}$ to $\mathscr{D}\mathfrak{L}$. This investigation needs suitable notions of interpolation and Beth definability for default logics. We show both positive and negative general results: depending on how $\mathscr{D}\mathfrak{L}$ is defined and of the kind of interpolation/Beth definability involved, the property might or might not transfer from $\mathfrak{L}$ to $\mathscr{D}\mathfrak{L}$.

1 Introduction

Interpolation and Beth definability are recognized as important properties of the meta-theory of a logic (see, e.g., [19]). Interpolation goes back to the seminal work of Craig in [11] and is, in one form, the following result: suppose that $\varphi \vdash \psi$, there is ξ in the common language of φ and ψ s.t. $\varphi \vdash \xi$ and $\xi \vdash \psi$. In addition to its theoretical relevance, interpolation has also proven to be influential in applications in Computer Science, e.g., in the context of software specification [6,14,25,34], in the construction of Formal Ontologies [23], and in Model Checking [26]. Though interpolation stands as a property in its own right, its main importance lies in the fact that it can be used to prove a result known as Beth definability via a standard argument (see, e.g., [28]). Intuitively, Beth definability implies that the syntax of the language is powerful enough to define any notion that is semantically fixed in a model. This is commonly regarded as a sign of a well behaved logic, where syntax and semantics are in harmony. Interpolation and Beth definability have received a lot less attention in non-classical and, in particular, non-monotonic logics. With this as our motivation, we investigate interpolation and Beth definability in default logics, a sub-class of the field of Non-monotonic Logic.

We start by defining a general framework which is sufficiently abstract to encompass most of the usual default logics (e.g., those introduced in [29,24,13,27]), generalizing ideas presented in [17,18]. We define a default logic $\mathscr{D}\mathfrak{L}$ on a base, monotonic, logic $\mathfrak{L}$ satisfying some minimal requirements. Then, we turn to the question of when interpolation and Beth definability results transfer from $\mathfrak{L}$ to $\mathscr{D}\mathfrak{L}$. As a result of the generality of our framework, we are able to prove far-reaching transfer results for a comprehensive class of default logics. We draw attention to the fact that interpolation and Beth definability for default logics needs suitable definitions. When dealing with a non-monotonic logical consequence relation $\mathrel{|\!\sim}$, it may not simply be possible to define interpolation as: if $\varphi \mathrel{|\!\sim} \psi$, then there is ξ in the common language of φ and ψ s.t. $\varphi \mathrel{|\!\sim} \xi$ and $\xi \mathrel{|\!\sim} \psi$. For starters, since $\mathrel{|\!\sim}$ is non-monotonic, it may not be transitive. Moreover, since consequence in most default logics is defined in terms of *default theories*, the notion of "common language", and the left and right hand sides of $\mathrel{|\!\sim}$ should also be dealt with care. After discussing how to define interpolation and Beth definability in default logics, we show both positive and negative results. Depending on how $\mathscr{D}\mathfrak{L}$ is defined and of which kind of interpolation/Beth definability property we study, the property might or might not transfer from $\mathfrak{L}$ to $\mathscr{D}\mathfrak{L}$. In particular, we show that the Strong Craig Interpolation Property (SCIP) always transfer from $\mathfrak{L}$ to $\mathscr{D}\mathfrak{L}$ (Prop. 6), while the Split Interpolation Property (SIP) fails for any traditional $\mathscr{D}\mathfrak{L}$ based on $\mathfrak{L}$ extending classical propositional logic (CPL), even though CPL has SIP (Prop. 7). Similarly, if $\mathfrak{L}$ has SIP and $\mathscr{D}\mathfrak{L}$ is stable under substitutions, then sceptical default consequence in $\mathscr{D}\mathfrak{L}$ has a version of the Beth definability property (Prop. 8), while this property fails for credulous default consequence in traditional $\mathscr{D}\mathfrak{L}$ based on $\mathfrak{L}$ extending CPL (Prop. 9).

Structure. In Sec. 2 we provide a general definition of a default logic. We start by defining what we require of a base logic in Sec. 2.1. We introduce default logics in Sec. 2.2, and define traditional default logics in Sec. 2.3. In Sec. 2.4 we briefly discuss strongly saturated default logics – a class of well behaved default logics generalizing traditional default logics. Sec. 3 investigates interpolation and Beth definability. We introduce appropriate definitions in Sec. 3.1 and Sec. 3.3. Our main results are shown in Sec. 3.2 and Sec. 3.4. Sec. 4 concludes the paper discussing related work and providing pointers for future research.

2 What is a Default Logic?

Default logics are a sub-class of non-monotonic logics. Different default logics have been introduced after the originating proposal in Reiter's seminal work [29]. These different default logics have in common the notion of a *default* and an *extension*. A default is a triple of formulas of a formal language, notation $\pi \overset{\rho}{\Rightarrow} \chi$, capturing a conditional, defeasible statement. An extension is a set of formulas making precise some constraints on π and ρ, enabling us to detach χ from $\pi \overset{\rho}{\Rightarrow} \chi$. Default logics differ from one another in the conditions enabling detaching a default. Defaults and extensions are basic ingredients in what is a default logic.

2.1 Preliminary Definitions

We define default logics over a *base logic*. In our setting, a base logic, or a *logic* for short, has two ingredients: a set of *formulas* and a *consequence* relation. Formulas are defined over a *language*, i.e., a triple $\mathscr{F} = \langle \mathscr{A}, \mathscr{L}, \mathscr{G} \rangle$ where: $\mathscr{A}$ is a set of non-logical symbols (the alphabet); $\mathscr{L}$ is a set of logical symbols with corresponding arities; and $\mathscr{G}$ are the rules of grammar. We also use $\mathscr{F}$ for the set of all formulas of a language $\mathscr{F}$. As usual, lowercase and uppercase Greek letters are variables for formulas and sets of formulas, resp. We restrict our attention to *propositional languages*, i.e., languages where $\mathscr{A}$ is a set of proposition symbols. We use p, q, r, etc., for proposition symbols. For any $\mathscr{F}$ and $\Phi \subseteq \mathscr{F}$, $\mathscr{A}(\Phi)$ is the *alphabet* of Φ, i.e., the set of proposition symbols appearing in formulas in Φ. We say that Φ is defined on an alphabet A if $\mathscr{A}(\Phi) \subseteq A$. We define $\mathscr{F}\restriction_A = \{ \varphi \in \mathscr{F} \mid \mathscr{A}(\varphi) \subseteq A \}$. We use $\mathrm{S}^p_q(\Phi)$ to indicate the result of substituting every appearance of p by q in every formula in Φ. A consequence relation $\vdash$ is a subset of $2^{\mathscr{F}} \times \mathscr{F}$ indicating what follows from what in a logic. We use $\Phi \vdash \varphi$ for $(\Phi, \varphi) \in \vdash$; and $\vdash \varphi$ if $\Phi = \emptyset$ (we omit brackets for singleton sets). We make no assumptions regarding whether $\vdash$ is defined syntactically or semantically. We do assume that $\vdash$ satisfies *reflexivity*, *monotonicity*, *cut*, and *structurality* (see, e.g., [16]). We make precise what a logic is in the next definition.

Definition 1 (Logic). *A logic is a tuple $\mathfrak{L} = \langle \mathscr{F}, \vdash \rangle$ where $\mathscr{F}$ is a language, and $\vdash \subseteq 2^{\mathscr{F}} \times \mathscr{F}$ is a consequence relation s.t. $\varphi \vdash \varphi$ (reflexivity); if $\Phi \vdash \varphi$ and $\Phi \subseteq \Phi'$, then $\Phi' \vdash \varphi$ (monotonicity); if $\Phi \vdash \varphi_i$ and $\Phi \cup \{ \varphi_i \mid i \in I \} \vdash \psi$, then $\Phi \vdash \psi$ (cut); and if $\Phi \vdash \varphi$, then $\mathrm{S}^p_q(\Phi) \vdash \mathrm{S}^p_q(\varphi)$ (structurality).*

For any logic $\mathfrak{L}$, we say that φ is a consequence of Φ iff $\Phi \vdash \varphi$. We define $\Phi^\bullet = \{ \varphi \mid \Phi \vdash \varphi \}$. The operator $(\_)^\bullet$ is a *closure operator*, i.e.: $\Phi \subseteq \Phi^\bullet$; if $\Phi \subseteq \Phi'$, then $\Phi^\bullet \subseteq \Phi'^\bullet$; and $\Phi^\bullet = \Phi^{\bullet\bullet}$. A set of sentences Φ is *consistent* if $\Phi^\bullet \subset \mathscr{F}$.

Definition 2. *An* implicative logic *is a logic whose logical symbols contain nullary symbols* $\top$ (verum) *and* $\bot$ (falsum)*, and a binary symbol* $\supset$ (implication)*; whose set $\mathscr{F}$ of formulas contains $\{\top \supset \varphi, \varphi \supset \top, \bot \supset \varphi, \varphi \supset \bot, \varphi \supset \psi\}$; and whose consequence relation satisfies: $\top \in \Phi^\bullet$ iff $\bot \supset \bot \in \Phi^\bullet$* ($\top\bot$-def)*; $\varphi \in \Phi^\bullet$ iff $\top \supset \varphi \in \Phi^\bullet$* ($\top$-left-neutral)*; if $\{\varphi \supset \phi, \phi \supset \psi\} \subseteq \Phi^\bullet$, $\varphi \supset \psi \in \Phi^\bullet$* ($\supset$-transitive)*; and if $\{\varphi, \varphi \supset \psi\} \subseteq \Phi^\bullet$, $\psi \in \Phi^\bullet$* (modus ponens)*.*

Henceforth, by a logic, we mean an implicative logic. Implicative logics play a fundamental role in our treatment of interpolation and Beth definability.

Example 1. The following are some typical cases of logics: Classical Propositional Logic (CPL) [15]; Intuitionistic Propositional Logic (IPL) [33]; the class of Normal Modal Logics [7]; in particular, the Basic Modal Logic K with *local* consequence [7]; the Basic Modal Logic K with *global* logical consequence [7]; the Modal Logic KAlt_1 [7]; the Standard Deontic Logic D [35,10]; the Deontic Logic KDA [9]; the epistemic logic $\mathsf{S5}$ [21,10]; and the hybrid logic $\mathsf{H}(\mathsf{A}, \downarrow)$ [5] (which is equivalent to Classical First-Order Logic over the appropriate language).

2.2 Default Logics

We start with a general definition of a default logic.

Definition 3 (Default Logic). *A default logic is a pair $\mathscr{D}\mathfrak{L} = \langle \mathfrak{L}, \mathscr{E} \rangle$ where $\mathfrak{L}$ is a logic and $\mathscr{E} : (2^{\mathscr{F}} \times 2^{(\mathscr{F}^3)}) \to 2^{(2^{\mathscr{F}})}$ is a function s.t. for every $E \in \mathscr{E}(\Phi, \Delta)$, $E = (\Phi \cup \{ \chi \mid (\pi, \rho, \chi) \in \Delta' \})^\bullet$ for some $\Delta' \subseteq \Delta$.*

$\mathscr{D} = \mathscr{F}^3$ is the set of all *defaults* of a default logic. $\pi \overset{\rho}{\Rightarrow} \chi$ is notation for $(\pi, \rho, \chi) \in \mathscr{D}$. A default theory Θ is a pair (Φ, Δ) where $\Phi \subseteq \mathscr{F}$ and $\Delta \subseteq \mathscr{D}$. If Θ is a default theory, Φ_Θ and Δ_Θ are the sets of formulas and defaults of Θ, resp. For a default theory Θ, $\mathscr{E}(\Theta)$ is its set of *extensions*. We associate with each default logic two notions of default consequence: *credulous*, and *sceptical*. Formally, φ is a credulous default consequence of a default theory Θ, notation $\Theta \mathrel{|\!\sim}^{\mathsf{c}} \varphi$, iff $\varphi \in \bigcup \mathscr{E}(\Theta)$; in turn, φ is a sceptical default consequence of Θ, notation $\Theta \mathrel{|\!\sim}^{\mathsf{s}} \varphi$, iff $\varphi \in \bigcap \mathscr{E}(\Theta)$. If $\mathscr{E}(\Theta) = \emptyset$, $\bigcup \mathscr{E}(\Theta) = \emptyset$ and $\bigcap \mathscr{E}(\Theta) = \mathscr{F}$ (see [32]). Define $\Theta^{\mathsf{c}} = \{ \varphi \mid \Theta \mathrel{|\!\sim}^{\mathsf{c}} \varphi \}$ and $\Theta^{\mathsf{s}} = \{ \varphi \mid \Theta \mathrel{|\!\sim}^{\mathsf{s}} \varphi \}$. We use $\mathrel{|\!\sim}$ and Θ^{d} when there is no need to distinguish between $\mathrel{|\!\sim}^{\mathsf{c}}$ and $\mathrel{|\!\sim}^{\mathsf{s}}$, and Θ^{c} and Θ^{s}, resp.

The rest of this section illustrates how some of the most common properties of Default Logics fit into our definition. We say that a default logic $\mathscr{DL}$ *guarantees extensions* iff for all Θ, $\mathscr{E}(\Theta) \neq \emptyset$. Default logics that guarantee extensions are *supra-classical*, i.e., for all Θ, $(\Phi_\Theta)^\bullet \subseteq \Theta^{\mathsf{d}}$; and they satisfy $\Theta^{\mathsf{s}} \subseteq \Theta^{\mathsf{c}}$ for all Θ. These properties are not satisfied if extensions fail to exist, i.e., if there is Θ s.t. $\mathscr{E}(\Theta) = \emptyset$. Let Θ_1 and Θ_2 be default theories, define $\Theta_1 \sqsubseteq \Theta_2$ iff $\Phi_{\Theta_1} \subseteq \Phi_{\Theta_2}$ and $\Delta_{\Theta_1} \subseteq \Delta_{\Theta_2}$. We say that $\mathscr{DL}$ is *non-monotonic* iff there are Θ_1 and Θ_2 s.t. $\Theta_1 \sqsubseteq \Theta_2$ and $(\Theta_1)^{\mathsf{d}} \nsubseteq (\Theta_2)^{\mathsf{d}}$. We say that $\mathscr{DL}$ is *semi-monotonic* iff for any two Θ_1 and Θ_2 s.t. $\Theta_1 \sqsubseteq \Theta_2$, if $\Phi_{\Theta_1} = \Phi_{\Theta_2}$, then for all $E_1 \in \mathscr{E}(\Theta_1)$, there is $E_2 \in \mathscr{E}(\Theta_2)$ s.t. $E_1 \subseteq E_2$. Further, we say that $\mathscr{DL}$ is *$\mathscr{E}$-consistent* iff for all Θ, if Φ_Θ is $\mathfrak{L}$-consistent, then all $E \in \mathscr{E}(\Theta)$ are $\mathfrak{L}$-consistent. Non-monotonocity, semi-monotonicity, and $\mathscr{E}$-consistency do not follow from Def. 3. Moreover, they need not be satisfied by default logics (even if they guarantee extensions); they depend on the particularities of the definition of $\mathscr{E}$. We make no assumptions regarding whether an arbitrary default logic satisfies any of the properties above.

2.3 Traditional Default Logics

Def. 3 paints a general picture of what is a default logic. At the same time, it captures default logics that are, in a sense, "degenerate". E.g., we can define a default logic s.t. for all Θ, $\mathscr{E}(\Theta) = \{(\Phi_\Theta)^\bullet\}$. This default logic ignores defaults, thus collapsing default reasoning into reasoning in the underlying logic, i.e., $\Theta^{\mathsf{c}} = \Theta^{\mathsf{s}} = (\Phi_\Theta)^\bullet$ for all Θ. We call any default logic satisfying this condition *trivial*. Trivial default logics are extreme cases of little interest from a Default Logic perspective. In defining a default logic, we wish to provide a precise account of what does it mean to reason with defaults in a way such that reasoning in the underlying logic is extended non-monotonically. This is the purpose of *traditional* default logics. Traditional default logics encompass Reiter's seminal work on default logic [29] and some of its major variants, e.g., [24,30,13,27], summarized in [2,12]. We introduce what we mean by a traditional default logic in Def. 8 by building on, and generalizing, definitions and results presented in [17,18].

We begin by taking a closer look at defaults. Typically, a default $\pi \overset{\rho}{\Rightarrow} \chi$ is intuitively read as: if π is *grounded* in what is known and ρ is *coherent* with what is known, then, *detach* χ and assume it tentatively as part of what is known. Extensions formalize the set of "known things", what is meant by grounded, coherent, and detached and assumed tentatively. How these concepts are formalized separate traditional default logics from each other, as different intuitions lead to different formalizations.

Henceforth, by *consistency* we mean $\mathfrak{L}$-consistency. Define, for all sets Δ, $\Delta^{\Pi} = \{ \pi \mid \pi \overset{\rho}{\Rightarrow} \chi \in \Delta \}$, $\Delta^{\mathrm{P}} = \{ \rho \mid \pi \overset{\rho}{\Rightarrow} \chi \in \Delta \}$ and $\Delta^{\mathrm{X}} = \{ \chi \mid \pi \overset{\rho}{\Rightarrow} \chi \in \Delta \}$.

Definition 4 (Grounded). *Let Θ be a default theory, and $\Delta_1 \subseteq \Delta_2 \subseteq \Delta_\Theta$; we say that Δ_2 is* grounded *in Δ_1 iff $\Delta_2^{\Pi} \subseteq (\Phi_\Theta \cup \Delta_1^{\mathrm{X}})^\bullet$. In addition, for all $\Delta \subseteq \Delta_\Theta$, we say that Δ is a closed set iff $\Delta^{\Pi} \subseteq (\Phi_\Theta \cup \Delta^{\mathrm{X}})^\bullet$.*

Def. 4 captures a standard view on what does it mean for a set of defaults to be grounded. Intuitively, if we think of the sets Δ_1 and Δ_2 as defaults "already considered" and defaults "to be considered", resp., the view of grounded in Def. 4 permits only for the consequents of "already considered" defaults to be used to establish the prerequisites of "to be considered" defaults. Closed sets are sets of defaults whose prerequisites can be established from within the set.

Definition 5 (Coherence). *Let Θ be a default theory, and $\Delta_1 \subseteq \Delta_2 \subseteq \Delta_\Theta$; we say that Δ_2 is* i-coherent *w.r.t. Δ_1 iff:*

(1-coherent) *for all $\delta_2 \in \Delta_2$, $(\Phi_\Theta \cup \Delta_1^{\mathrm{X}} \cup \delta_2^{\mathrm{P}})^\bullet$ is consistent.*
(2-coherent) *for all $\delta_2 \in \Delta_2$, $(\Phi_\Theta \cup \Delta_2^{\mathrm{X}} \cup \delta_2^{\mathrm{P}})^\bullet$ is consistent.*
(3-coherent) *$(\Phi_\Theta \cup \Delta_1^{\mathrm{X}} \cup \Delta_2^{\mathrm{P}})^\bullet$ is consistent.*
(4-coherent) *$(\Phi_\Theta \cup \Delta_2^{\mathrm{X}} \cup \Delta_2^{\mathrm{P}})^\bullet$ is consistent.*

In addition, we say that Δ_2 is self *i*-coherent *if it is i-coherent w.r.t. itself.*

Proposition 1. *i-coherence implies 1-coherence, while 4-coherence implies i-coherence, for $1 \leq i \leq 4$. Further, self 1-coherence implies self 2-coherence; self 3-coherence implies self 4-coherence.*

A default $\pi \overset{\rho}{\Rightarrow} \chi \in \Delta$ is *normal* iff $\rho = \chi$. We use $\pi \Rightarrow \chi$ as notation for normal defaults. A default theory Θ is *normal* if all $\delta \in \Delta_\Theta$ are normal.

Proposition 2. *For normal default theories, the four notions of coherence introduced in Def. 5 are equivalent.*

Def. 5 captures four different views on what does it mean for a set of defaults to be coherent. The relation between these different views is made clear in Props. 1 and 2. Again, if we think of the sets Δ_1 and Δ_2 as defaults "already considered" and defaults "to be considered", resp., 1-coherence and 2-coherence require the justifications of the defaults "to be considered" to be individually consistent w.r.t. the defaults "already considered". They differ from each other in whether or not the consequents of the defaults "to be considered" should be included in the consistency check. These takes on coherence correspond to Reiter [29,24] and to Łukaszewicz [24], resp. In turn, 3-coherence and 4-coherence require the justifications of the defaults "to be considered" to be jointly consistent; and differ from each other in whether or not the consequents of the defaults "to be considered" should be included in the consistency check. These takes on coherence correspond to Mikitiuk and Truszczynski [27], and to Delgrande, Jackson, and Schaub [30,13], resp. When there is no need to distinguish between the different types of coherence, we simply say that Δ_2 is *coherent* w.r.t. Δ_1.

Table 1. Coherence and Detachment

Coherence	Detachment	Proponent	Reference
1-coherence	classical	Reiter	[29]
2-coherence	justified	Łukaszewicz	[24]
3-coherence	rational	Mikitiuk and Truszczynski	[27]
4-coherence	constrained	Delgrande, Jackson, and Schaub	[13,30]

Definition 6 (Detachment). *Let Θ be a default theory and $\Delta_1, \Delta_2 \subseteq \Delta_\Theta$; we say that Δ_2 is* detached *by Δ_1 if Δ_2 is grounded in, and coherent w.r.t., Δ_1. We say that δ is detached by Δ_1 if $\Delta_2 = \Delta_1 \cup \delta$ is detached by Δ_1.*

Intuitively, detachment can be thought of as a version of modus-ponens for defaults. Fixing a definition of coherence, we say that detachment is: *classical*, *justified*, *rational*, and *constrained*, according to Table 1.

Remark 1. Recall that every well-ordering $\prec$ is order-equivalent to exactly one ordinal number τ. Such an ordinal number τ is the order type of $\prec$. The precise definitions of these terms, and that of a limit ordinal, can be found in [32].

Definition 7. *Let Θ be a default theory; we say that $\Delta \subseteq \Delta_\Theta$ is* regular *if there is a well-ordering $\prec$ on Δ_Θ s.t. $\Delta = \mathsf{D}^{\prec}_{\Theta}(\tau)$, where τ is the order type of $\prec$, and for all ordinals ω s.t. $0 < \omega < \tau$, and all limit ordinals λ s.t. $\lambda \leq \tau$, $\mathsf{D}^{\prec}_{\Theta}$ is defined:*

$$\mathsf{D}^{\prec}_{\Theta}(0) = \emptyset$$

$$\mathsf{D}^{\prec}_{\Theta}(\omega+1) = \begin{cases} \mathsf{D}^{\prec}_{\Theta}(\omega) \cup \delta & \textit{if } \delta \in (\Delta \setminus \mathsf{D}^{\prec}_{\Theta}(\omega)) \textit{ is detached by } \mathsf{D}^{\prec}_{\Theta}(\omega) \textit{ and for all other} \\ & \delta' \in (\Delta \setminus \mathsf{D}^{\prec}_{\Theta}(\omega)), \textit{ if } \delta' \textit{ is detached by } \mathsf{D}^{\prec}_{\Theta}(\omega),\ \delta \prec \delta' \\ \mathsf{D}^{\prec}_{\Theta}(\omega) & \textit{otherwise} \end{cases}$$

$$\mathsf{D}^{\prec}_{\Theta}(\lambda) = \bigcup \{\, \mathsf{D}^{\prec}_{\Theta}(\omega) \mid \omega \leq \lambda \,\}$$

Again, Def. 7 encompasses four kinds of regularity. We say that a regular set of defaults is: *classical*, *justified*, *rational*, and *constrained*, depending on the definition of detachment it uses. Regularity captures a prescriptive view of how to cumulatively detach defaults in default theory. The function $\mathsf{D}^{\prec}_{\Theta}$ is the closure under detachment of a set of defaults under the selection strategy defined by the well-ordering $\prec$, and is a standard definition of a function by *transfinite* recursion. Ex. 2 illustrates the need for transfiniteness.

Example 2. In some cases, we may wish to prove that our default logic is semi-monotonic. Suppose that extensions in a default logic $\mathscr{DL}$ are obtained through regular sets of defaults, and only those sets. This example shows that unless we allow for transfinite steps, we may fail to prove semi-monotonicity due to restrictions on the definition of $\mathsf{D}^{\prec}_{\Theta}$. Let $\mathscr{DL}$ be an $\mathscr{E}$-consistent default logic built over $\mathsf{KAlt}_1\Diamond^+$ (i.e., the modal logic where $\Diamond$ is interpreted over a weakly functional accessibility relation, and $\Diamond^+$ is its transitive closure, see, e.g., [7]). Let Θ_1 be a default theory s.t. $\Phi_{\Theta_1} = \{\Diamond\top\}$ and $\Delta_{\Theta_1} = \{\, \Diamond^i\top \Rightarrow \Diamond^{(i+1)}\top \mid i \geq 0 \,\}$. Δ_{Θ_1} is regular for all kinds of detachment. Let $E_1 = (\Phi_{\Theta_1} \cup \Delta_{\Theta_1})^\bullet = \{ \Diamond^i \top \mid i \geq 0 \}^\bullet$. E_1 is satisfied in Kriple models in which every world has a successor. Let Θ_2 be s.t. $\Theta_1 \sqsubseteq \Theta_2$, $\Phi_{\Theta_1} = \Phi_{\Theta_2}$, and $\Delta_{\Theta_2} = \Delta_{\Theta_1} \cup \{\top \Rightarrow \Diamond^+ \Box \bot\}$; the formula $\Diamond^+ \Box \bot$ describes the existence of a world reachable in a finite number of steps through the accessibility relation, which has no successors. There is no well-ordering $\prec$ on Δ_{Θ_2} of order type ω_0 s.t. $E_1 \subseteq (\{\Diamond \top\} \cup (\mathsf{D}^{\prec}_{\Theta_2}(\omega_0))^{\mathrm{X}})^\bullet$. To see why, note that any such $\prec$ on Δ_{Θ_2} contains $\top \Rightarrow \Diamond^+ \Box \bot$ at some position n. Since $\top \Rightarrow \Diamond^+ \Box \bot$ is detached by any $\Delta_2 \subseteq \Delta_{\Theta_2}$, $\mathsf{D}^{\prec}_{\Theta_2}(\omega_0)$ detaches $\top \Rightarrow \Diamond^+ \Box \bot$ in at most n steps. But as soon as $\top \Rightarrow \Diamond^+ \Box \bot$ is detached no other default in Δ_{Θ_2} can be detached. Thus, for all $\prec$, $E_2 = (\{\Diamond \top\} \cup (\mathsf{D}^{\prec}_{\Theta_2}(\omega_0))^{\mathrm{X}})^\bullet$ is satisfied in Kripke models consisting of chains of at most n worlds; and so $E_1 \nsubseteq E_2$. This establishes a failure of semi-monotonicity. By allowing transfinite steps, we can first detach all defaults in Δ_{Θ_1}, and then proceed to check whether or not $\top \Rightarrow \Diamond \Box \bot$ can be detached in a transfinite step. From this, we can recover semi-monotonicity.

Definition 8 (Traditional Default Logic). *We say that a default logic $\mathscr{DL}$ is* traditional *iff for all default theories Θ, $\mathscr{E}(\Theta)$ is the smallest set s.t. for all regular and self coherent subsets Δ of Δ_Θ, there is $E \in \mathscr{E}(\Theta)$ s.t. $E = (\Phi_\Theta \cup \Delta^{\mathrm{X}})^\bullet$.*

From Def. 8 it is possible to prove that traditional default logics encompass four distinct sub-classes of default logics. These classes are: classical default logics (classical regularity and 1-coherence); justified default logics (justified regularity and 2-coherence); rational default logics (rational regularity and 3-coherence); and constrained default Logics (constrained regularity and 4-coherence). This claim is made precise in prop. 3

Proposition 3. *Every traditional default logic is either a classical, a justified, a constrained, or a rational default logic, and vice-versa.*

It follows by construction, adapting the argument in [17,18], that *Classical Default Logic*, defined by Reiter in [29], is a classical default logic. The same is true, *mutatis mutandis*, for *Justified Default Logic*, defined by Łukaszewicz in [24], *Rational Default Logic*, defined by Mikitiuk and Truszczynski in [27], and *Constrained Default Logic*, defined by Delgrande, Jackson, and Schaub in [13].

2.4 Intermediate Default Logics

Def. 3 paints a very general picture of what is a default logic. In turn, Def. 8 captures default logics whose extensions are obtained in a very prescriptive way via regular set of defaults. The obvious question is whether there are some "interesting" default logics "stricter" than those in Def. 3 but "weaker" than those in Def. 8.

Definition 9. *Let $\mathscr{DL}$ be any default logic, and Θ be any default theory; we say that $\Delta \subseteq \Delta_\Theta$ is saturated iff for all $\Delta' \subseteq \Delta_\Theta$, if Δ' is detached by Δ, $\Delta' \subseteq \Delta$.*

Definition 10. *We say that a default logic $\mathscr{DL}$ is* weakly saturated *iff for all default theories* Θ *and all* $E \in \mathscr{E}(\Theta)$*, there exists a saturated* $\Delta \subseteq \Delta_\Theta$ *s.t.* $E = (\Phi_\Theta \cup \Delta^{\mathrm{X}})^\bullet$*. In addition, we say that* $\mathscr{DL}$ *is* strongly saturated *iff it is weakly saturated and for all default theories* Θ *and all saturated* $\Delta \subseteq \Delta_\Theta$*, if* Δ *is self coherent, then there is* $E \in \mathscr{E}(\Theta)$ *s.t.* $E = (\Phi_\Theta \cup \Delta^{\mathrm{X}})^\bullet$*.*

Proposition 4. *Every traditional default logic is strongly saturated.*

We can think of weakly saturated default logics as imposing an "upper bound" on extensions, i.e., anything that is not a saturated set of defaults cannot be an extension. On the other hand, strongly saturated default logics impose a "lower bound" on extensions, i.e., anything that is a saturated and self coherent set of defaults must be an extension. Strongly saturated default logics are an interesting generalization of traditional default logics for they simplify the proof of some results circumventing the prescriptive definition of extensions in Def. 8.

3 Interpolation and Beth Definability

As mentioned, interpolation and Beth definability are recognized as important properties of the meta-theory of a logic. Here, we investigate interpolation and Beth definability in Default Logics. More precisely, we investigate when results transfer from $\mathfrak{L}$ to $\mathscr{DL}$. In order to accomplish this, we first need to formulate suitable notions of interpolation and Beth definability for a default logics.

3.1 Interpolation

There is no unifying definition of interpolation in the literature, see [22]. Instead, this property comes in many flavours. In what follows, we discuss some relevant formulations of interpolation. In this discussion we assume an arbitrary logic $\mathfrak{L}$.

Let us start with the so-called *Craig Interpolation Property* (CIP).

Definition 11. *We say that consequence in* $\mathfrak{L}$ *has* CIP *iff whenever* $\vdash \varphi \supset \psi$*, there is* ξ *defined on* $\mathscr{A}(\varphi) \cap \mathscr{A}(\psi)$ *s.t.* $\vdash \varphi \supset \xi$ *and* $\vdash \xi \supset \psi$*.*

On certain occasions, in place for CIP, we may wish to have a stronger version.

Definition 12. *We say that consequence* $\mathfrak{L}$ *has the* Strong Craig Inteprolation Property (SCIP) *iff whenever* $\Phi \vdash \varphi \supset \psi$*, there is* ξ *defined on* $\mathscr{A}(\Phi, \varphi) \cap \mathscr{A}(\psi)$ *s.t.* $\Phi \vdash \varphi \supset \xi$ *and* $\Phi \vdash \xi \supset \psi$*.*

A rather different formulation of interpolation, used in the standard argument for *Beth Definability*, is the so-called *Split Interpolation Property* (SIP), see [31].

Definition 13. *We say that consequence* $\mathfrak{L}$ *has* SIP *iff for any* Φ *and* φ *defined on an alphabet* A_1*, and any* Ψ *and* ψ *defined on an alphabet* A_2*; if* $\Phi \cup \Psi \vdash \varphi \supset \psi$*, there is* ξ *defined on* $A_1 \cap A_2$ *s.t.* $\Phi \vdash \varphi \supset \xi$ *and* $\Psi \vdash \xi \supset \psi$*. The formula* ξ *is called a* split interpolant*.*

In general, CIP, SCIP, and SIP are not equivalent (having one does not imply having the others). Equivalence depends on the particularities of the logical connectives under consideration and on logical consequence satisfying properties such as compactness, deduction, etc. Logics known to have all three different versions of interpolation are, for example, CPL, IPL, and the modal logics K, $\mathsf{S5}$ and $\mathsf{H}(\mathsf{A}, \downarrow)$ with local and global consequence. For a discussion regarding equivalence of interpolation in these logics see [4,3]. We take a particular interest in SIP: as an interpolation result in its own right, given its widespread applicability, and as a step towards obtaining Beth definability in a standard way [28].

3.2 Interpolation in Default Logics

We explore what the natural formulations of CIP, SCIP, and SIP, look like for default consequence in default logics.

Definition 14. *We say that default consequence in a default logic $\mathscr{D}\mathfrak{L}$ has the* Default Craig Interpolation Property*, notation $\mathscr{D}\mathsf{CIP}$, iff whenever $\mathrel{|\!\sim} \varphi \supset \psi$, there is ξ defined on $\mathscr{A}(\varphi) \cap \mathscr{A}(\psi)$, s.t. $\mathrel{|\!\sim} \varphi \supset \xi$ and $\mathrel{|\!\sim} \xi \supset \psi$.*

Proposition 5. *For any default logic $\mathscr{D}\mathfrak{L} = \langle \mathfrak{L}, \mathscr{E} \rangle$; if consequence in $\mathfrak{L}$ has CIP, then, default consequence in $\mathscr{D}\mathfrak{L}$ has $\mathscr{D}\mathsf{CIP}$.*

The proof of Prop. 5 is direct from the definition of a default logic and CIP for $\vdash$ in $\mathfrak{L}$. $\mathscr{D}\mathsf{CIP}$ is rather trivial as it involves only reasoning from empty default theories, thus reducing default consequences to consequences in the underlying logic. Let us consider the more interesting case of $\mathscr{D}\mathsf{SCIP}$, the SCIP version of interpolation for Default Logics, which makes use of non-empty default theories.

Definition 15. *We say that default consequence in a default logic $\mathscr{D}\mathfrak{L}$ has the* Default Strong Craig Interpolation Property*, notation $\mathscr{D}\mathsf{SCIP}$, iff whenever $\Theta \mathrel{|\!\sim} \varphi \supset \psi$, there is ξ defined on $\mathscr{A}(\Theta, \varphi) \cap \mathscr{A}(\psi)$, s.t. $\Theta \mathrel{|\!\sim} \varphi \supset \xi$ and $\Theta \mathrel{|\!\sim} \xi \supset \psi$.*

Proposition 6. *For any default logic $\mathscr{D}\mathfrak{L} = \langle \mathfrak{L}, \mathscr{E} \rangle$; if consequence in $\mathfrak{L}$ has SCIP, then, default consequence in $\mathscr{D}\mathfrak{L}$ has $\mathscr{D}\mathsf{SCIP}$.*

Proof (by cases). Let Θ be a default theory; if $\mathscr{E}(\Theta) = \emptyset$, $\Theta^{\mathsf{c}} = \emptyset$ and $\Theta^{\mathsf{s}} = \mathscr{F}$. The result follows trivially from these facts. If $\mathscr{E}(\Theta) \neq \emptyset$:

(c) Let $\Theta \mathrel{|\!\sim}^{\mathsf{c}} \varphi \supset \psi$; then, there is $E \in \mathscr{E}(\Theta)$ s.t. $E \vdash \varphi \supset \psi$. From SCIP, there is ξ defined on $\mathscr{A}(E, \varphi) \cap \mathscr{A}(\psi)$ s.t. $E \vdash \varphi \supset \xi$ and $E \vdash \xi \supset \psi$. So, $\Theta \mathrel{|\!\sim}^{\mathsf{c}} \varphi \supset \xi$ and $\Theta \mathrel{|\!\sim}^{\mathsf{c}} \xi \supset \psi$, with $\mathscr{A}(\xi) \subseteq \mathscr{A}(\Theta, \varphi) \cap \mathscr{A}(\psi)$.

(s) Let $\Theta \mathrel{|\!\sim}^{\mathsf{s}} \varphi \supset \psi$, and $\Gamma = \bigcap \mathscr{E}(\Theta)$; then, $\Gamma \vdash \varphi \supset \psi$. From SCIP, there is ξ defined on $\mathscr{A}(\Gamma, \varphi) \cap \mathscr{A}(\psi)$ s.t. $\Gamma \vdash \varphi \supset \xi$ and $\Gamma \vdash \xi \supset \psi$. Thus, $\Theta \mathrel{|\!\sim}^{\mathsf{s}} \varphi \supset \xi$ and $\Theta \mathrel{|\!\sim}^{\mathsf{s}} \xi \supset \psi$, with $\mathscr{A}(\xi) \subseteq \mathscr{A}(\Theta, \varphi) \cap \mathscr{A}(\psi)$.

The result follows from (c) and (s).

We now turn our attention to what does SIP look like for default consequence.

Remark 2. For default theories Θ_i, define $\Theta_1 \sqcup \Theta_2 = (\Phi_{\Theta_1} \cup \Phi_{\Theta_2}, \Delta_{\Theta_1} \cup \Delta_{\Theta_2})$.

Definition 16. *We say that default consequence in a default logic $\mathscr{D}\mathfrak{L}$ has the* Default Split Interpolation Property*, notation $\mathscr{D}$SIP, iff for all Θ_1 and φ defined on an alphabet A_1, and Θ_2 and ψ defined on an alphabet A_2, if $\Theta_1 \sqcup \Theta_2 \mathrel{|\!\sim} \varphi \supset \psi$, there is ξ defined on $A_1 \cap A_2$, s.t. $\Theta_1 \mathrel{|\!\sim} \varphi \supset \xi$ and $\Theta_2 \mathrel{|\!\sim} \xi \supset \psi$.*

For $\mathscr{D}$SIP we obtain a negative result in the following form.

Proposition 7. *For any traditional default logic $\mathscr{D}\mathfrak{L}$ built on a logic extending Classical Propositional Logic, default consequence in $\mathscr{D}\mathfrak{L}$ does not have $\mathscr{D}$SIP.*

Proof. W.l.o.g. let $\mathfrak{L}$ be CPL, and $\Theta_1 = (\{p\}, \{p \Rightarrow q\})$ and $\Theta_2 = (\emptyset, \{q \Rightarrow r\})$, it follows that:

(1) for all $E_1 \in \mathscr{E}(\Theta_1)$, $E_1 = \{p, q\}^\bullet$.
(2) for all $E_2 \in \mathscr{E}(\Theta_2)$, $E_2 = \emptyset^\bullet$.
(3) for all $E \in \mathscr{E}(\Theta_1 \sqcup \Theta_2)$, $E_3 = \{p, q, r\}^\bullet$.

From (3), $\Theta_1 \sqcup \Theta_2 \mathrel{|\!\sim} p \supset r$. Immediately, $p \in \mathscr{A}(\Theta_1)$, and $r \in \mathscr{A}(\Theta_2)$. Then, any formula ξ defined on $\mathscr{A}(\Theta_1) \cap \mathscr{A}(\Theta_2)$ is equivalent to $\top$, $\bot$, q, or $\neg q$. If we fix ξ to any of these formulas, either from (1), $\Theta_1 \not\mathrel{|\!\sim} p \supset \xi$; or from (2), $\Theta_2 \not\mathrel{|\!\sim} \xi \supset r$.

We explored some natural formulations of CIP, SCIP, and SIP for default consequence in a default logic. We have shown positive transfer results for $\mathscr{D}$CIP and $\mathscr{D}$SCIP. We highlight the generality of these results: not only they concern traditional logics, but all default logics. This level of generality, i.e., proofs depending on extensions and not their construction, is achieved thanks to the abstract presentation of what is a default logic. We have also shown a negative transfer result for $\mathscr{D}$SIP. In this case the counter-example is much more concrete, but still sufficiently general to cover all traditional default logics. Lack of $\mathscr{D}$SIP is a set back for Beth definability, as we are now pre-empted to use the standard argument for establishing the latter from the former [28]. Nonetheless, we show that Beth definability can still be obtained in some form for some default logics.

3.3 Definability

Beth definability is commonly regarded as a sign of a well behaved logic. We adapt our definition of this property from [22].

Definition 17. *Let $\mathfrak{L}$ be any logic, Φ be a set of formulas s.t. $\mathscr{A}(\Phi) \subseteq A$, and $q \notin A$; we say that consequence in $\mathfrak{L}$ has the* Beth Definability Property (BDP) *iff whenever*

$$\Phi \cup \mathrm{S}^p_q(\Phi) \vdash p \supset q \quad \text{and} \quad \Phi \cup \mathrm{S}^p_q(\Phi) \vdash q \supset p \tag{1}$$

there is ε defined on an alphabet $A_0 = A \setminus \{p\}$ s.t.

$$\Phi \vdash p \supset \varepsilon \quad \text{and} \quad \Phi \vdash \varepsilon \supset p \tag{2}$$

Eq. (1) *expresses that Φ* implicitly defines *p; whereas Eq.* (2) *is the* explicit definition *of p from Φ.*

In general, BDP can be obtained from SIP through a standard argument [28]. Let us remark that failure of SIP does not necessarily imply failure of BDP. The latter property may still be obtained through other means.

3.4 Definability in Default Logics

Def. 18 introduces a natural formulation of Beth definability for default logics.

Definition 18. *Let* $\mathscr{D}\mathfrak{L}$ *be a default logic,* Θ *a default theory s.t.* $\mathscr{A}(\Theta) \subseteq A$*, and* $q \notin A$*; we say that default consequence in* $\mathscr{D}\mathfrak{L}$ *has the* Default Beth definability property ($\mathscr{D}$BDP) *iff whenever*

$$\Theta \sqcup \mathrm{S}^p_q(\Theta) \mathrel{|\!\sim} p \supset q \quad \textit{and} \quad \Theta \sqcup \mathrm{S}^p_q(\Theta) \mathrel{|\!\sim} q \supset p \tag{3}$$

there is ε *defined on an alphabet* $A_0 = A \setminus \{p\}$ *s.t.*

$$\Theta \mathrel{|\!\sim} p \supset \varepsilon \quad \textit{and} \quad \Theta \mathrel{|\!\sim} \varepsilon \supset p \tag{4}$$

Eq. (3) *expresses that* Θ implicitly defines p*; whereas* ε *in Eq.* (4) *is the* explicit definition *of* p *from* Θ*.*

Proving Beth definability for default consequence in default logics requires some additional definitions and lemmas (the proofs of which are in App. A). First, it needs a condition on *stability*, see Def. 19. This condition states that the extensions of a default theory are in harmony with the extensions of its extended default theory under substitution.

Definition 19. *We say that a default logic* $\mathscr{D}\mathfrak{L} = \langle \mathfrak{L}, \mathscr{E} \rangle$ *is* stable *iff for all default theories* Θ *defined on an alphabet* A*, if* $q \notin A$*, it follows that for all* $E \in \mathscr{E}(\Theta)$*, there is* $E' \in \mathscr{E}(\Theta \sqcup \mathrm{S}^p_q(\Theta))$ *s.t.* $E' = (E \cup \mathrm{S}^p_q(E))^\bullet$*.*

Lemma 1 shows that the condition of being stable is rather natural, in the sense that it is satisfied by a non-trivial class of default logics, i.e., those that are strongly saturated and, in particular, by traditional default logics.

Lemma 1. *Any strongly saturated default logic* $\mathscr{D}\mathfrak{L}$ *is stable.*

Lemma 2 establishes that the notion coherence for the extensions of a given default theory is preserved if we augment the default theory by substitution.

Lemma 2. *Let* $\mathscr{D}\mathfrak{L}$ *be a default logic; for all default theories* Θ *and all* $\Delta \subseteq \Delta_\Theta$*, if* Δ *is self coherent in* Θ*, then,* $\Delta \cup \mathrm{S}^p_q(\Delta)$ *is self coherent in* $\Theta \sqcup \mathrm{S}^p_q(\Theta)$*.*

The following lemma, simplifies a key step in the proof of Prop. 8.

Lemma 3. *Let* $\{\, \Phi_i \mid i \in I \,\}$ *be a set of sets of formulas s.t. for all* $i \in I$*,* $\mathscr{A}(\Phi_i) \subseteq A$ *and* $\Phi_i = {\Phi_i}^\bullet$*; if consequence in* $\mathfrak{L}$ *has* SIP*,* $q \notin A$*, and* $p \supset q \in \bigcap \{\, (\Phi_i \cup \mathrm{S}^p_q(\Phi_i))^\bullet \;\big|\; i \in I \,\}$*, then* $p \supset q \in (\bigcap \{\, \Phi_i \mid i \in I \,\} \cup \bigcap \{\, \mathrm{S}^p_q(\Phi_i) \;\big|\; i \in I \,\})^\bullet$*.*

Proposition 8. *For any default logic* $\mathscr{D}\mathfrak{L} = \langle \mathfrak{L}, \mathscr{E} \rangle$*; if* $\mathscr{D}\mathfrak{L}$ *is stable and consequence in* $\mathfrak{L}$ *has* SIP*, then, sceptical default consequence in* $\mathscr{D}\mathfrak{L}$ *has* $\mathscr{D}$BDP*.*

Proof (by cases). Let Θ be any default theory defined on alphabet A, and $q \notin A$; if $\mathscr{E}(\Theta \sqcup \mathrm{S}^p_q(\Theta)) = \emptyset$, the result holds trivially from the fact that $\mathscr{D}\mathfrak{L}$ is stable. Otherwise, i..e, if $\mathscr{E}(\Theta \sqcup \mathrm{S}^p_q(\Theta)) \neq \emptyset$, let $\Theta \sqcup \mathrm{S}^p_q(\Theta) \mathrel{|\!\!\sim}^{\mathsf{s}} p \supset q$; from the fact that $\mathscr{D}\mathfrak{L}$ is stable, $p \supset q \in \bigcap\{ (E \cup \mathrm{S}^p_q(E))^\bullet \mid E \in \mathscr{E}(\Theta) \}$. From Lemma 3, $p \supset q \in (\bigcap\{ E \mid E \in \mathscr{E}(\Theta) \} \cup \bigcap\{ \mathrm{S}^p_q(E) \mid E \in \mathscr{E}(\Theta) \})^\bullet$. From SIP, there is ξ defined on $A \setminus \{p\}$ s.t. $p \supset \xi \in \bigcap\{ E \mid E \in \mathscr{E}(\Theta) \}$ and (†) $\xi \supset q \in \bigcap\{ \mathrm{S}^p_q(E) \mid E \in \mathscr{E}(\Theta) \}$. Substituting p for q in (†) we obtain $\xi \supset p \in \bigcap\{ E \mid E \in \mathscr{E}(\Theta) \}$. Therefore, there is ξ defined on $A \setminus \{p\}$ s.t. $\Theta \mathrel{|\!\!\sim}^{\mathsf{s}} p \supset \xi$ and $\Theta \mathrel{|\!\!\sim}^{\mathsf{s}} \xi \supset p$.

Corollary 1. *For all traditional default logics built on a logic $\mathfrak{L}$; if $\mathfrak{L}$ has* SIP*, then sceptical consequence has $\mathscr{D}$*BDP*.*

For credulous default consequence we obtain the following negative result.

Proposition 9. *For all traditional default logic $\mathscr{D}\mathfrak{L}$ built on a logic extending* CPL*, credulous default consequence in $\mathscr{D}\mathfrak{L}$ does not have $\mathscr{D}$*SIP*.*

Proof. W.l.o.g. let $\mathfrak{L}$ be CPL; consider a default theory $\Theta = (\emptyset, \{\delta_1, \delta_2\})$, where

$$\delta_1 = \top \overset{\neg p}{\Longrightarrow} [(\neg p \vee r) \wedge s] \qquad\qquad \delta_2 = s \overset{p}{\Rightarrow} (p \wedge \neg r)$$

Trivially, we get $\mathrm{S}^p_q(\Theta) = (\emptyset, \{\top \overset{\neg q}{\Longrightarrow} [(\neg q \vee r) \wedge s], s \overset{q}{\Rightarrow} (q \wedge \neg r)\})$. Moreover:

(1) In classical, justified, constrained, and rational default logic on $\mathfrak{L}$, it follows that, $\mathscr{E}(\Theta \sqcup \mathrm{S}^p_q(\Theta)) \supseteq \{ (\Delta_i^{\mathrm{X}})^\bullet \mid i \in \{1,2\} \}$ where: $\Delta_1 = \{\delta_1, \mathrm{S}^p_q(\delta_2)\}$; and $\Delta_2 = \{\mathrm{S}^p_q(\delta_1), \delta_2\}$.
(2) In justified, constrained, and rational default logic on $\mathfrak{L}$, it follows that, for all $E \in \mathscr{E}(\Theta)$, $E = (\{\delta_1\}^{\mathrm{X}})^\bullet$.
(3) In classical default logic on $\mathfrak{L}$, it follows that, $\mathscr{E}(\Theta) = \emptyset$.

Clearly, $q \notin \mathscr{A}(\Theta)$. From (1), $\Theta \sqcup \mathrm{S}^p_q(\Theta) \mathrel{|\!\!\sim}^{\mathsf{c}} p \supset q$ and $\Theta \sqcup \mathrm{S}^p_q(\Theta) \mathrel{|\!\!\sim}^{\mathsf{c}} q \supset p$. To see why, note that $(\Delta_1^{\mathrm{X}})^\bullet = \{[(\neg p \vee r) \wedge s], \neg r\}^\bullet$ and $(\Delta_2^{\mathrm{X}})^\bullet = \{[(\neg q \vee r) \wedge s], \neg r\}^\bullet$ are both in $\mathscr{E}(\Theta \sqcup \mathrm{S}^p_q(\Theta))$. Immediately, $\{[(\neg p \vee r) \wedge s], \neg r\} \vdash p \supset q$, and also $\{[(\neg q \vee r) \wedge s], \neg r\} \vdash q \supset p$. In justified, constrained, and rational default logic on $\mathfrak{L}$, there is no ξ defined on $\mathscr{A}(\Theta) \setminus \{p\}$ for which $\Theta \mathrel{|\!\!\sim}^{\mathsf{c}} p \supset \xi$ and $\Theta \mathrel{|\!\!\sim}^{\mathsf{c}} \xi \supset p$. To see why, note from (2) that every $E \in \mathscr{E}(\Theta)$ is equal to $\{(\neg p \vee r) \wedge s\}^\bullet$. Let E be any such extension, it is easy to see that there are models $\mathfrak{M}_1$ and $\mathfrak{M}_2$ of E s.t. $\mathfrak{M}_1 \Vdash p$ and $\mathfrak{M}_1 \Vdash \neg p$. This establishes failure of $\mathscr{D}$BDP for justified, constrained, and rational default logic on $\mathfrak{L}$. In classical default logic on $\mathfrak{L}$, there is no ξ defined on $\mathscr{A}(\Theta) \setminus \{p\}$ for which $\Theta \mathrel{|\!\!\sim}^{\mathsf{c}} p \supset \xi$ and $\Theta \mathrel{|\!\!\sim}^{\mathsf{c}} \xi \supset p$ simply because $\mathscr{E}(\Theta) = \emptyset$. This establishes failure of $\mathscr{D}$BDP for classical default logic on $\mathfrak{L}$. In summary, the default theory Θ defined above exhibits a counter-example for $\mathscr{D}$BDP for credulous default consequence in any traditional default logic built on a logic extending CPL.

Even though $\mathscr{D}$SIP fails for default logics, we showed that under certain conditions, $\mathscr{D}$BDP can be still obtained for the sceptical default consequence.

4 Final Remarks

Interpolation and Beth definability are recognized as important properties of the meta-theory of a logic. However, few authors have explored these properties in the field of Non-monotonic Logic, and in default logics in particular. A pioneering work in this area is [1]. Therein the author studies interpolation for circumscription, default logic, and logic programs with the stable models semantics. The version of interpolation presented in [1] is different from the ones investigated here, and is proven for sceptical default consequence in what we would call classical default logic over CPL (with finite vocabularies). The author also formulates a version of split interpolation and proves it for credulous consequence in the same context. However, the proof of this property requires the alphabet of the consequences of one default theory to be disjoint from the alphabet of the prerequisites and justifications of the defaults in other default theory. Thus the result applies to a restricted set of cases. In contrast, our results hold for a richer collection of default logics and generalize some of those introduced in [1]. Another interesting interpolation result in the field of Non-monotonic is [20]. This work studies interpolation in *equilibrium logic*; presenting a technique to obtain interpolation results by relying on the fact that the version of non-monotonic consequence in question can be defined via some minimally (axiomatically) defined models in some monotonic logic. This technique does not directly apply in default logics, since minimal sets of models of the base logics are not immediately connected to extensions. But this deserves a deeper investigation. We are, to the best of our knowledge, unaware of investigations of Beth definability in default logics.

We investigated interpolation and Beth definability in default logics. To this end, we started with a presentation of a general frawework for defining a default logic $\mathscr{D}\mathfrak{L}$ from a basic monotonic logic $\mathfrak{L}$. This framework covers well-known traditional default logics found in the literature, but encompasses a much richer family of default logics. Then, we defined suitable versions of interpolation and Beth definability for Default Logics, and studied their statuses. Given the generality of our definition of a default logic, the discussed results hold (or fail to hold) for several versions of default logics. In particular, we showed that CIP and SCIP (two versions of the so-called *Craig Interpolation Property*) transfers from $\mathfrak{L}$ to $\mathscr{D}\mathfrak{L}$, but Split Interpolation SIP fails for default logics extending CPL, even if $\mathfrak{L}$ has it. When considered as a step towards Beth definability, this negative result is a set back. However, we showed that the sceptical default consequence in a $\mathscr{D}\mathfrak{L}$ has Beth definability ($\mathscr{D}$BDP) if $\mathscr{D}\mathfrak{L}$ is stable (i.e., the extensions of a default theory are in harmony with those of its augmented default theory under substitution) and $\mathfrak{L}$ has SIP. Different is the case for credulous default consequence, in which $\mathscr{D}$BDP fails for any $\mathscr{D}\mathfrak{L}$ built on a logic extending CPL.

We view this work as a first step towards a better understanding of the meta-theory of default logics in general. As future work, it would be interesting to apply similar ideas to study proof calculi for default logics that are parameterized on the underlying logic. Moreover, it would be interesting to see whether the methods for constructing interpolants in the underlying proof calculi transfer to the default version (see e.g. [8]).

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A Selected Proofs

Remark 3. Let $\mathfrak{L}$ be any logic, and Φ and Ψ be sets of sentences; we say that Ψ is a *conservative extension* of Φ, notation $\Psi \geq \Phi$, iff $\Phi^\bullet \subseteq \Psi^\bullet$ and $(\Psi^\bullet\restriction_{\mathscr{A}(\Phi)}) \subseteq \Phi^\bullet$.

Lemma 4. *Let $\mathfrak{L}$ be any logic, Φ be a set of sentences defined on an alphabet A, and $q \notin A$; if consequence in $\mathfrak{L}$ has* SIP*, $\Phi \cup \mathrm{S}^p_q(\Phi) \geq \Phi$ and $\Phi \cup \mathrm{S}^p_q(\Phi) \geq \mathrm{S}^p_q(\Phi)$.*

Proof. Trivially, $\Phi^\bullet \subseteq (\Phi \cup \mathrm{S}^p_q(\Phi))^\bullet$. In turn, let $\varphi \in (\Phi \cup \mathrm{S}^p_q(\Phi))^\bullet\restriction_{\mathscr{A}(\Phi)}$; then, $\Phi \cup \mathrm{S}^p_q(\Phi) \vdash \varphi$, alt., $\Phi \cup \mathrm{S}^p_q(\Phi) \vdash \top \supset \varphi$. From SIP, there is ε defined on $A \setminus \{p\}$ s.t. $\mathrm{S}^p_q(\Phi) \vdash \top \supset \varepsilon$ and $\Phi \vdash \varepsilon \supset \varphi$. Since $q \notin \mathscr{A}(\Phi, \varepsilon)$, $\mathrm{S}^q_p(\mathrm{S}^p_q(\Phi)) \vdash \mathrm{S}^q_p(\top \supset \varepsilon)$ results in $\Phi \vdash \top \supset \varepsilon$. Then, $\Phi \vdash \top \supset \varepsilon$ and $\Phi \vdash \varepsilon \supset \varphi$; and so, $\Phi \vdash \top \supset \varphi$, alt., $\Phi \vdash \varphi$. Therefore, $(\Phi \cup \mathrm{S}^p_q(\Phi))^\bullet\restriction_{\mathscr{A}(\Phi)} \subseteq \Phi^\bullet$.

Lemma 1. *Any strongly saturated default logic $\mathscr{D}\mathfrak{L}$ is stable.*

Proof. Let Θ be a default theory defined on an alphabet A, and and $q \notin A$. In addition, let $E \in \mathscr{E}(\Theta)$ be s.t. $E = (\Phi_\Theta \cup \Delta^{\mathrm{X}})^\bullet$ for some $\Delta \subseteq \Delta_\Theta$. Since $\mathscr{D}\mathfrak{L}$ is strongly saturated, Δ is saturated in Θ. The result follows immediately if $\Delta \cup \mathrm{S}^p_q(\Delta)$ is saturated in $\Theta \sqcup \mathrm{S}^p_q(\Theta)$; as $E' = (\Phi_\Theta \cup \mathrm{S}^p_q(\Phi_\Theta) \cup (\Delta \cup \mathrm{S}^p_q(\Delta))^{\mathrm{X}})^\bullet$ is our extension. The proof proceeds by contradiction. Let $\Delta \cup \mathrm{S}^p_q(\Delta)$ be not saturated in $\Theta \sqcup \mathrm{S}^p_q(\Theta)$; w.l.o.g. there is a default $\delta \notin \Delta \cup \mathrm{S}^p_q(\Delta)$ s.t. δ is detached by $\Delta \cup \mathrm{S}^p_q(\Delta)$. Clearly, $\delta \in \Delta_\Theta$ or $\delta = \mathrm{S}^p_q(\delta')$ for some $\delta' \in \Delta_\Theta$. If $\delta \in \Delta_\Theta$, from Lemma 4, δ is detached by Δ; and so Δ is not saturated. This yields a contradition. If $\delta = \mathrm{S}^p_q(\delta')$ for some $\delta' \in \Delta_\Theta$, from Lemma 4, $\mathrm{S}^p_q(\delta')$ is detached by $\mathrm{S}^p_q(\Delta)$; and so $\mathrm{S}^p_q(\Delta)$ is not saturated. But by substitution, δ' is detached by Δ, and so Δ is not saturated. This also yields a contradiction. Thus, $\Delta \cup \mathrm{S}^p_q(\Delta)$ is saturated in $\Theta \sqcup \mathrm{S}^p_q(\Theta)$.

Lemma 2. *Let $\mathscr{D}\mathfrak{L}$ be a default logic; for all default theories Θ and all $\Delta \subseteq \Delta_\Theta$, if Δ is self coherent in Θ, then, $\Delta \cup \mathrm{S}^p_q(\Delta)$ is self coherent in $\Theta \sqcup \mathrm{S}^p_q(\Theta)$.*

Proof. Similar to that of Lemma 1

Lemma 3. *Let $\{\, \Phi_i \mid i \in I \,\}$ be a set of sets of formulas s.t. for all $i \in I$, $\mathscr{A}(\Phi_i) \subseteq A$ and $\Phi_i = {\Phi_i}^\bullet$; if consequence in $\mathfrak{L}$ has* SIP*, $q \notin A$, and $p \supset q \in \bigcap\{\, (\Phi_i \cup \mathrm{S}^p_q(\Phi_i))^\bullet \bigm| i \in I \,\}$, then $p \supset q \in (\bigcap\{\, \Phi_i \mid i \in I \,\} \cup \bigcap\{\, \mathrm{S}^p_q(\Phi_i) \bigm| i \in I \,\})^\bullet$.*

Proof (by contradiction). Let us assume that $p \supset q \in \bigcap\{\, (\Phi_i \cup \mathrm{S}^p_q(\Phi_i))^\bullet \mid i \in I \,\}$; by definition, it follows that all $(*)$ $\Phi_i \cup \mathrm{S}^p_q(\Phi_i) \vdash p \supset q$. At the same time, let $p \supset q \notin (\bigcap\{\, \Phi_i \mid i \in I \,\} \cup \bigcap\{\, \mathrm{S}^p_q(\Phi_i) \bigm| i \in I \,\})^\bullet$; then, for all ξ defined on $A \setminus \{p\}$, either $(\dagger)$ $p \supset \xi \notin \bigcap\{\, \Phi_i \mid i \in I \,\}$ or $(\ddagger)$ $\xi \supset q \notin \bigcap\{\, \mathrm{S}^p_q(\Phi_i) \bigm| i \in I \,\}$. From $(\dagger)$, there is $\Phi_i \nvdash p \supset \xi$; and from Lemma 4, $(\S)$ $\Phi_i \cup \mathrm{S}^p_q(\Phi_i) \nvdash p \supset \xi$. But $(\S)$ leads to a contradiction; since from $(*)$ $\Phi_i \cup \mathrm{S}^p_q(\Phi_i) \vdash p \supset q$, by SIP, there is in fact ξ defined on $A \setminus \{p\}$ s.t. $\Phi_i \cup \mathrm{S}^p_q(\Phi_i) \vdash p \supset \xi$! Similarly, we obtain a contradiction from $(\ddagger)$. Thus, $p \supset q \in (\bigcap\{\, \Phi_i \mid i \in I \,\} \cup \bigcap\{\, \mathrm{S}^p_q(\Phi_i) \bigm| i \in I \,\})^\bullet$.