Deontic Action Logics via Algebra

Abstract

Deontic logics are dubbed the logics of normative or prescriptive reasoning. These logics can roughly be categorized into *ought-to-be*, dealing with the prescription of state of affairs, or *ought-to-do*, dealing with the prescription of actions. An important family of *ought-to-do* deontic logics have their origin in Segerberg's *Deontic Action Logic* (DAL, see [23]). In this work, we provide an algebraic characterization of DAL and some known variants. In brief, we capture actions and formulas as elements of different base algebras, and deontic operators as algebraic operations; different algebras capture the different variants. This algebraization enables us to obtain completeness results via standard algebraic means. Moreover, we argue that this algebraic framework offers a natural way of (re-)thinking many deontic logical issues at large.

Keywords: Deontic Action Logic, Algebraic Logic, Normative Reasoning.

1 Introduction

Deontic Logic (DL) is devoted to the study of norms and their logical foundations. The beginnings of DL can be traced back to the pioneer works of G. von Wright [28], J. Kalinowski [13], and O. Becker [5]. Since then, most deontic logics have been defined as particular classes of modal logics (see [7,6]). The most famous among these formal systems is *Standard Deontic Logic*, SDL for short. SDL extends the normal modal system K with the extra axiom D for *seriality*. An in-depth introduction to diverse formal systems of deontic logic is provided in [4], together with a historical presentation.

Deontic logics built on SDL are known as *ought-to-be*, as they deal with the prescription of states of affairs, i.e., propositions. However, G. von Wright pointed out that deontic logics are closely related to the concept of action (see [28,29]), and furthermore, they should be constructed upon a theory of actions (see [28,29]). These observations, also shared by other authors (e.g., [14,23,19,9,8,26,21]), have led to the development of deontic logics where prescriptions apply to actions instead of propositions. Deontic logics of this kind are called *ought-to-do*.

One of the first *ought-to-do* deontic logics was presented by K. Segerberg in [23]. Segerberg's logic formally distinguishes between actions and formulas. In this formalism, actions are built up from basic action names using action combinators. Then, deontic connectives apply to actions to yield formulas, and formulas are obtained from formulas using logical connectives. We illustrate this by means of a simple example. Let driving and drinking be basic action names; the formula $\neg\mathsf{P}(\mathsf{driving} \sqcap \mathsf{drinking})$ states that drinking while driving is not permitted. In this formula, $\sqcap$ is an action operator that can be understood as the parallel execution of actions; P is the deontic connective of permission, and $\neg$ is logical negation. The obtained logic is extremely simple and admits a sound and complete proof system. An interesting feature of Segerberg's logic is its two tier interpretation structure, i.e., actions are interpreted resorting to an algebra of events, whereas formulas are interpreted using truth values. Segerberg's initial formalism was revisited by other authors, for instance: [9] introduces action prescriptions and combines them with modal operators, and [25] investigates several fragments of Segerberg's logic. We follow the terminology from [25] and call these formalisms *deontic action logics*.

In this paper we provide an algebraic formulation of deontic action logics. More precisely, we develop an abstract view of deontic action logics in terms of algebraic structures. To this end, we follow some of the main ideas introduced by Halmos in [11], where Boolean algebras serve as an abstraction of propositions; Venema in [27], who introduced Boolean algebras with operators as an algebraic counterpart of modal logics; and Pratt in [20], who introduced dynamic algebras to investigate the theoretical properties of dynamic logics via many-sorted algebras. Intuitively, in our framework, formulas are captured as elements of a Boolean algebra, while actions are formalized by means of another (Boolean) algebra. In this setting, deontic operators are modeled as functions connecting both algebras. We put forth that the benefits of this algebraic version of deontic action logics are twofold. Firstly, algebraic logic has been shown useful when analyzing theoretical properties of logics and investigating the relations between different formalisms. Secondly, extensions to a deontic action logic can be obtained by considering different action and predicate algebras. We explore these ideas in Sec. 4.

Structure. In Sec. 2, we introduce some of the basic definitions of Segerberg's deontic action logic, called DAL. In Sec. 3, we present the basic algebraic framework, and prove an algebraic version of soundness and completeness for DAL using standard algebraic tools. Preliminary definitions about algebra used in that section can be found in Appendix A. In Sec. 4, we discuss variants of deontic action logics using particular classes of algebras. Lastly, in Sec. 5, we offer some final remarks and discuss future work.

2 Segerberg's Deontic Action Logic

We cover the syntax and semantics of the deontic action logic originally introduced by Segerberg in [23]. We refer to this logic as DAL.

Syntax of DAL. The language of DAL is comprised of a set Act of *actions* and a set Form of *formulas* defined on a countable set $\mathsf{Act}_0 = \{ \mathsf{a}_i \mid i \in \mathbb{N} \}$ of basic action symbols. The sets Act and Form are given by the grammars in Eq. (1) and Eq. (2), respectively:

$$\alpha ::= \mathsf{a}_i \mid \alpha \sqcup \alpha \mid \alpha \sqcap \alpha \mid \bar{\alpha} \mid \mathsf{0} \mid \mathsf{1} \tag{1}$$

$$\varphi ::= \mathsf{P}\alpha \mid \mathsf{F}\alpha \mid \alpha = \beta \mid \varphi \to \varphi \mid \neg\varphi. \tag{2}$$

Intuitively, any $\mathsf{a}_i \in \mathsf{Act}_0$ is a *basic* action; $\alpha \sqcup \beta$ is the *free-choice* between α and β; $\alpha \sqcap \beta$ is the *parallel* execution of α and β; $\bar{\alpha}$ is the *complement* of α, i.e., any action other than α; and $\mathsf{0}$ and $\mathsf{1}$ are the *impossible* and the *universal* actions, respectively. Turning to formulas, the connective $=$ indicates *equality* of actions. The logical connectives $\rightarrow$ and $\neg$ stand for *material implication* and *negation*, respectively. We also consider the derived logical connectives: $\vee$ for *disjunction*, $\wedge$ for *conjunction*, $\top$ for *verum*, $\bot$ for *falsum*, and $\leftrightarrow$ for *material bi-implication*. The derived logical connectives are defined from $\rightarrow$ and $\neg$ in the usual way. The connectives P and F have a deontic reading: (a) P stands for *permitted*, i.e., α is allowed to be executed; (b) F stands for *forbidden*, i.e., the execution of α forbidden.

The axioms for DAL are listed in Fig. 1. A Hilbert-style notion of provability based on these axioms is defined in the usual way using the rule of *modus ponens*. More precisely, a proof of φ is a finite sequence $\psi_1, \ldots, \psi_n$ of formulas s.t. $\psi_n = \varphi$, and for each $k \leq n$, ψ_k is either: (i) an axiom; or (ii) obtained from two earlier formulas using *modus ponens*, i.e., there are $i, j < k$ s.t. $\psi_j = \psi_i \rightarrow \psi_k$. We say that φ is a theorem of DAL, written $\vdash \varphi$, if there is a proof of φ. The set of theorems of DAL is the set: $\{\varphi \mid \vdash \varphi\}$.

1. The following set of axioms for actions α, β, and γ (see [10]):

(3)	$\alpha \sqcup \mathsf{0} = \alpha$	(4)	$\alpha \sqcap \mathsf{1} = \alpha$
(5)	$\alpha \sqcup \mathsf{1} = \mathsf{1}$	(6)	$\alpha \sqcap \mathsf{0} = \mathsf{0}$
(7)	$\alpha \sqcup \beta = \beta \sqcup \alpha$	(8)	$\alpha \sqcap \beta = \beta \sqcap \alpha$
(9)	$\alpha \sqcup (\beta \sqcap \gamma) = (\alpha \sqcup \beta) \sqcap (\alpha \sqcup \gamma)$	(10)	$\alpha \sqcap (\beta \sqcup \gamma) = (\alpha \sqcap \beta) \sqcup (\alpha \sqcap \gamma)$

2. The following set of axioms for formulas φ, ψ, χ (see [18]):

(11) $\varphi \rightarrow (\psi \rightarrow \varphi)$
(12) $(\neg\varphi \rightarrow \neg\psi) \rightarrow ((\neg\varphi \rightarrow \psi) \rightarrow \varphi)$
(13) $(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi))$

3. The following set of axioms for $(=)$:

(14) $\alpha = \alpha$ (15) $(\alpha = \beta) \rightarrow (\beta = \alpha)$ (16) $(\alpha = \beta) \wedge (\beta = \gamma) \rightarrow (\alpha = \gamma)$

4. The substitution axiom:

$$(17)\ (\alpha = \beta) \rightarrow (\varphi \rightarrow \varphi^{\beta}_{\alpha})$$

where φ^{β}_{α} is the formula obtained from replacing some ocurrences of α with β.

5. The deontic axioms:

(18) $\mathsf{P}(\alpha \sqcup \beta) \leftrightarrow (\mathsf{P}\alpha \wedge \mathsf{P}\beta)$ (19) $\mathsf{F}(\alpha \sqcup \beta) \leftrightarrow (\mathsf{F}\alpha \wedge \mathsf{F}\beta)$
(20) $(\mathsf{P}\alpha \wedge \mathsf{F}\alpha) \leftrightarrow (\alpha = \mathsf{0})$

Figure 1. Axioms for DAL

Semantics of DAL. A deontic action model is a tuple $\mathfrak{M} = \langle E, P, F \rangle$ where: (a) E is a set of elements; and (b) P and F are subsets of E satisfying $P \cap F =$

$\emptyset$. Intuitively, in a deontic action model $\mathfrak{M}$, we can think of the set E the set of possible outcomes of actions, and of the sets P and F as sets of permitted and forbidden events. The condition $P \cap F = \emptyset$ in (b) can be understood as an indication that: *only the impossible is both permitted and forbidden.*

A valuation on a deontic model $\mathfrak{M}$ is a function $v : \mathsf{Act}_0 \to 2^E$. Every valuation v extends uniquely to a function $v^* : \mathsf{Act} \to 2^E$ defined as

$$
\begin{aligned}
v^*(\alpha \sqcup \beta) &= v^*(\alpha) \cup v^*(\beta) \\
v^*(\alpha \sqcap \beta) &= v^*(\alpha) \cap v^*(\beta) \\
v^*(\bar{\alpha}) &= E \setminus v^*(\alpha) \\
v^*(0) &= \emptyset \\
v^*(1) &= E.
\end{aligned}
$$

The notion of satisfiability in a deontic action model under a valuation v, written $\mathfrak{M}, v \vDash \varphi$, is inductively defined as:

$$
\begin{aligned}
\mathfrak{M}, v \vDash \alpha = \beta &\iff v^*(\alpha) = v^*(\beta) \\
\mathfrak{M}, v \vDash \mathsf{P}\alpha &\iff v^*(\alpha) \subseteq P \\
\mathfrak{M}, v \vDash \mathsf{F}\alpha &\iff v^*(\alpha) \subseteq F \\
\mathfrak{M}, v \vDash \varphi \to \psi &\iff \mathfrak{M}, v \nvDash \varphi \text{ or } \mathfrak{M}, v \vDash \psi \\
\mathfrak{M}, v \vDash \neg\varphi &\iff \mathfrak{M}, v \nvDash \varphi.
\end{aligned}
$$

A formula φ is universally valid, written $\vDash \varphi$, iff for any deontic action model $\mathfrak{M}$ and valuation v on $\mathfrak{M}$, it follows that $\mathfrak{M}, v \vDash \varphi$.

3 DAL via Algebra

The logical formalism introduced by Segerberg in [23] enjoys some interesting characteristics. In particular, it is a simple modal logic that provides a well-executed characterization of deontic operators. Moreover, it enjoys an elegant semantics via ideals and Boolean algebras, or dually via sets and collections of sets. Furthermore, Segerberg's formalism further accommodates for additional deontic operators to be added sistematically. More importantly, the formalism is sound and complete (Theorem 3.1 in [23]).

In this section, we revise Segerberg's formalism from an algebraic perspective. More precisely, we provide an algebraic generalization of DAL. This generalization preserves the aforementioned properties of the original system. In particular, the algebraic theory is simple and uses standard tools of algebras (Boolean algebras, homomorphisms, free generated algebras, etc). It is modular in the sense that the algebras described below can be straightforwardly extended to support other deontic operators. And it also addresses the soundness and completeness of DAL using standard algebraic tools. It is worth remarking that the framework described below is, arguably, mathematically more abstract that the original DAL. This is one of the characteristics of algebraic logics which can be exploited to discuss some deontic logical issues at large. We retake this point later on.

3.1 Algebraic Background

In what follows, we assume that the reader is familiar with the following algebraic concepts. A (many-sorted) *signature* $\Sigma = \langle S, \Omega \rangle$ is a pair of a set S of sort names, or sorts, and a set Ω of function names. Each $f \in \Omega$ is assigned a non-empty sequence of elements of S indicating its type; formally: $\text{type}(f) = s_0 \dots s_n \to s$. A Σ-Algebra is a structure $\mathbf{A} = \langle \{A_s\}_{s \in S}, \{f_\mathbf{A}\}_{f \in F} \rangle$ where $f_\mathbf{A} : A_{s_0} \times \dots \times A_{s_n} \to A_s$ iff $\text{type}(f) = s_0 \dots s_n \to s$. Given a family of (mutually disjoint) sets of variables $X = \{X_s\}_{s \in S}$ and a signature Σ, $\mathbf{T}_\Sigma(X)$ denotes the term algebra constructed from Σ and X. An interpretation is a homomorphism $h : \mathbf{T}_\Sigma(X) \to \mathbf{A}$ which assigns meaning to the elements of $\mathbf{T}_\Sigma(X)$. A Σ-equation is a pair (t_1, t_2) of terms of $\mathbf{T}_\Sigma(X)$ written as $t_1 \approx t_2$. Given an algebra $\mathbf{A}$ and an interpretation h, we write $\mathbf{A}, i \vDash t_1 \approx t_2$ iff $h(t_1) = h(t_2)$. Moreover, we write $\mathbf{A} \vDash t_1 \approx t_2$ iff $\mathbf{A}, h \vDash t_1 \approx t_2$ holds for every interpretation h. We also assume some basics notions of Boolean algebras. Given a Boolean algebra $\mathbf{A}$, $\preccurlyeq_\mathbf{A}$ denotes its underlying partial order. An ideal is a lower subset of $\mathbf{A}$ w.r.t. $\preccurlyeq_\mathbf{A}$ closed under finite joins, and a filter is an upper subset of $\preccurlyeq_\mathbf{A}$ closed under finite meets. $\mathbf{2}$ is the Boolean algebra containing exactly two elements. A Boolean algebra is called *concrete* if it is a field of sets. We use Stone's representation theorem. In particular, for any Boolean algebra $\mathbf{A}$, $\mathsf{s}(\mathbf{A})$ denotes its isomorphic Stone space, and $\varphi_\mathbf{A} : \mathbf{A} \to \mathsf{s}(\mathbf{A})$ is the corresponding isomorphism. These and other useful notions are introduced in more detail in Appendix A.

3.2 Algebraizing DAL

One of the first steps in algebraizing a logic is to view formulas of a logical language as terms of an algebraic language. We begin by being clear about the algebraic language that we will use in the rest of this section.

Definition 3.1 The similarity type of DAL is a pair $\Sigma = (S, \Omega)$ where: (a) $S = \{a, f\}$ is a set of sort names and (b) $\Omega = \{\sqcup, \sqcap, ^-, 0, 1, \vee, \wedge, \neg, \bot, \top, =, \mathsf{P}, \mathsf{F}\}$ is a set of operation names s.t.:

(21) $\sqcup : a \times a \to a$	(22) $\sqcap : a \times a \to a$	(23) $^- : a \to a$	(24) $\mathsf{0} : a$
(25) $1 : a$	(26) $\vee : f \times f \to f$	(27) $\wedge : f \times f \to f$	(28) $\neg : f \to f$
(29) $\bot : f$	(30) $\top : f$	(31) $= : a \times a \to f$	(32) $\mathsf{P} : a \to f$
(33) $\mathsf{F} : a \to f$			

Intuitively, we think of the elements a and f of S in the signature Σ as sort names for actions and formulas, respectively. In turn, we think of the operation names in Ω as names for operators on actions, operators on formulas, or heterogeneous operators. The algebraic language we will use in the rest of this section is the freely generated algebra over the similarity type Σ of DAL w.r.t. the set Act_0 of basic action symbols. We refer to this algebra, written $\mathbf{T}$, as the *deontic action term algebra*.

Having defined the algebraic language, we turn our attention to the way in which this language is interpreted in an algebra. In this regard, just as Boolean algebras are fundamental for the algebraization of Classical Propositional Logic,

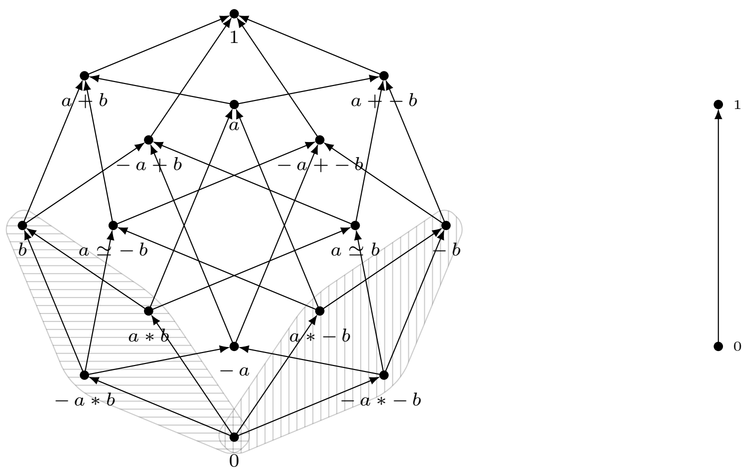

Figure 2. A Deontic Action Algebra

what we call *deontic action algebras* are fundamental for the algebraization of DAL. We introduce deontic action algebras in Def. 3.2 and discuss the technical details and the intuitions leading to this definition shortly after. (This notion borrows ideas and terminology from Pratt's *dynamic algebras* [20].)

Definition 3.2 A deontic action algebra is a tuple $\mathbf{D} = \langle \mathbf{A}, \mathbf{F}, \mathcal{E}, \mathcal{P}, \mathcal{F} \rangle$ where: (a) $\mathbf{A} = \langle A, +_{\mathbf{A}}, *_{\mathbf{A}}, -_{\mathbf{A}}, 0_{\mathbf{A}}, 1_{\mathbf{A}} \rangle$ and $\mathbf{F} = \langle F, +_{\mathbf{F}}, *_{\mathbf{F}}, -_{\mathbf{F}}, 0_{\mathbf{F}}, 1_{\mathbf{F}} \rangle$ are Boolean algebras; and (b) $\mathcal{E} : A \times A \to F$, $\mathcal{P} : A \to F$, and $\mathcal{F} : A \to F$, are total functions satisfying:

(34) $\mathcal{P}(a +_{\mathbf{A}} b) =_{\mathbf{F}} \mathcal{P}(a) *_{\mathbf{F}} \mathcal{P}(b)$ (35) $\mathcal{P}(a) *_{\mathbf{F}} \mathcal{F}(a) =_{\mathbf{F}} \mathcal{E}(a, 0_{\mathbf{A}})$

(36) $\mathcal{F}(a +_{\mathbf{A}} b) =_{\mathbf{F}} \mathcal{F}(a) *_{\mathbf{F}} \mathcal{F}(b)$ (37) $\mathcal{E}(a, b) *_{\mathbf{F}} \mathcal{P}(a) \preccurlyeq_{\mathbf{F}} \mathcal{P}(b)$

(38) $\mathcal{E}(a, b) *_{\mathbf{F}} \mathcal{F}(a) \preccurlyeq_{\mathbf{F}} \mathcal{F}(b)$ (39) $a =_{\mathbf{A}} b$ iff $\mathcal{E}(a, b) =_{\mathbf{F}} 1_{\mathbf{F}}$.

From an intuitive point of view, the elements in a deontic action algebra **D** may be understood as: (a) **A** and **F** correspond to an algebra of actions and an algebra of formulas, respectively; (b) $\mathcal{P}$ and $\mathcal{F}$ are abstract versions of the operations of an action being permitted and an action being forbidden, respectively; (c) $\mathcal{E}$ is an abstract version of the equality on actions at the level of formulas. From a technical point of view, Eq. (39) occupies a special place. This equation, in contrast to the others, is not expressed by an identity. Instead, it is expressed as a pair of conditional identities, or quasi-identities. This renders the class of deontic action algebras a quasi-variety (see [22]).

Definition 3.3 The quasi-variety of deontic action algebras is denoted by $\mathscr{D}_0$.

We give an example of a deontic action algebra $\mathbf{D} = \langle \mathbf{A}, \mathbf{F}, \mathcal{E}, \mathcal{P}, \mathcal{F} \rangle$ in Fig. 2. In this figure, the graph on the left illustrates the Boolean algebra **A** of actions. This algebra is the free Boolean algebra on the set of generators $\{a, b\}$. We use $x \simeq y$ as syntax sugar for $(x * y) + (-x * -y)$. The graph on the right illustrates the Boolean algebra **F** of formulas. This algebra is the

Boolean algebra **2**. We omitted subscripts on the operations of the Boolean algebras to improve legibility. The functions $\mathscr{P}$ and $\mathscr{F}$ are defined in Eqs. (40) and (41). The area shaded with horizontal lines illustrates the elements of $|\mathbf{A}|$ that $\mathscr{P}$ maps to 1, i.e., the elements of $|\mathbf{A}|$ that are permitted. Notice that these elements form an ideal in $\mathbf{A}$. In turn, the area shaded with vertical lines illustrates the elements of $|\mathbf{A}|$ that $\mathscr{F}$ maps to 1, i.e., the elements of $|\mathbf{A}|$ that are forbidden. Again, notice that these elements also form an ideal in $\mathbf{A}$. It can easily be seen in this example that: if $\mathscr{P}(x) = 1$ for all $x \in |\mathbf{A}|$, then, $\mathscr{F}(0) = 1$ and $\mathscr{F}(x) = 0$ for all $0 \neq x \in |\mathbf{A}|$. Similarly, if $\mathscr{F}(x) = 1$ for all $x \in |\mathbf{A}|$, then, $\mathscr{P}(0) = 1$ and $\mathscr{P}(x) = 0$ for all $0 \neq x \in |\mathbf{A}|$. These cases are known as *deontic heaven* and *deontic hell*, respectively. We will briefly discuss them later on.

$$(40)\ \mathscr{P}(x) = \begin{cases} 1 & \text{if } x \preccurlyeq b \\ 0 & \text{otherwise} \end{cases} \qquad (41)\ \mathscr{F}(x) = \begin{cases} 1 & \text{if } x \preccurlyeq -b \\ 0 & \text{otherwise} \end{cases}$$

We are now in a position to establish the connection between deontic action algebras and DAL.

Definition 3.4 Let $\mathbf{D}$ be a deontic algebra; an assignment on $\mathbf{D}$ is a function $f : \mathsf{Act}_0 \to |\mathbf{A}|$. An interpretation on $\mathbf{D}$ is a homomorphism $h : \mathbf{T} \to \mathbf{D}$ s.t.:

$$\begin{array}{lll} h(\alpha \sqcup \beta) = h(\alpha) +_{\mathbf{A}} h(\beta) & h(\phi \vee \psi) = h(\phi) +_{\mathbf{F}} h(\psi) & h(\mathsf{P}\alpha) = \mathscr{P}(h(\alpha)) \\ h(\alpha \sqcap \beta) = h(\alpha) *_{\mathbf{A}} h(\beta) & h(\phi \wedge \psi) = h(\phi) *_{\mathbf{F}} h(\psi) & h(\mathsf{F}\alpha) = \mathscr{F}(h(\alpha)) \\ h(\bar{\alpha}) = -_{\mathbf{A}}\, h(\alpha) & h(\neg\varphi) = -_{\mathbf{F}}\, h(\varphi) & h(\top) = 1_{\mathbf{F}} \\ h(\mathsf{0}) = 0_{\mathbf{A}} & h(\bot) = 0_{\mathbf{F}} & \\ h(\mathsf{1}) = 1_{\mathbf{A}} & h(\alpha = \beta) = \mathscr{E}(h(\alpha), h(\beta)) & \end{array}$$

Fact 3.5 *Assignments extend uniquely to interpretations. Given an assignment f, f^* denotes its unique extension.*

Definition 3.6 An equation is a pair (τ_1, τ_2), written $\tau_1 \approx \tau_2$, where either $\tau_1, \tau_2 \in \mathsf{Act}$ or $\tau_1, \tau_2 \in \mathsf{Form}$. An equation $\tau_1 \approx \tau_2$ is valid under an interpretation h on a deontic algebra $\mathbf{D}$, written $\mathbf{D}, h \models \tau_1 \approx \tau_2$, iff $h(\tau_1) = h(\tau_2)$. An equation $\tau_1 \approx \tau_2$ is universally valid, written $\models \tau_1 \approx \tau_2$, iff for all deontic algebras $\mathbf{D}$ and interpretations h on $\mathbf{D}$, it follows that $\mathbf{D}, h \models \tau_1 \approx \tau_2$.

Theorem 3.7 (Soundness) *If $\vdash \varphi$, then, $\models \varphi \approx \top$.*

Proof [Sketch] By induction on the length of a proof of φ. We restrict our attention to some interesting cases. In particular, to the axioms displayed in Eqs. (17), (18) and (20). Let $\mathbf{D}$ be any deontic algebra and h be any homomorphism on $\mathbf{D}$:

Eq. (17): We need to show that $h((\alpha = \beta) \to (\varphi \to \varphi^{\alpha}_{\beta})) = 1_{\mathbf{F}}$. The simple cases in which $\varphi = \mathsf{P}\alpha$ or $\varphi = \mathsf{F}\alpha$ entail all others. Then,

$$\begin{aligned} h((\alpha = \beta) \to (\mathsf{P}\alpha \to \mathsf{P}\beta)) &= h(\neg(\alpha = \beta) \vee (\neg\mathsf{P}\alpha \vee \mathsf{P}\beta)) \\ &= -_{\mathbf{F}}\, h(\alpha = \beta) +_{\mathbf{F}} h(\neg\mathsf{P}\alpha) +_{\mathbf{F}} h(\mathsf{P}\beta) \\ &= -_{\mathbf{F}}\, \mathscr{E}(h(\alpha), h(\beta)) +_{\mathbf{F}} -_{\mathbf{F}}\, h(\mathsf{P}\alpha) +_{\mathbf{F}} \mathscr{P}(h(\beta)) \\ &= -_{\mathbf{F}}\, \mathscr{E}(h(\alpha), h(\beta)) +_{\mathbf{F}} -_{\mathbf{F}}\, \mathscr{P}(h(\alpha)) +_{\mathbf{F}} \mathscr{P}(h(\beta)) \\ &= -_{\mathbf{F}}(\mathscr{E}(h(\alpha), h(\beta)) *_{\mathbf{F}} \mathscr{P}(h(\alpha))) +_{\mathbf{F}} \mathscr{P}(h(\beta)) \end{aligned}$$

From Eq. (37), $\mathscr{P}(h(\beta)) = \mathscr{E}(h(\alpha), h(\beta)) *_{\mathbf{F}} \mathscr{P}(h(\alpha)) +_{\mathbf{F}} \mathscr{P}(h(\beta))$. From this fact, $-_{\mathbf{F}}(\mathscr{E}(h(\alpha), h(\beta)) *_{\mathbf{F}} \mathscr{P}(h(\alpha))) +_{\mathbf{F}} \mathscr{P}(h(\beta)) = 1_{\mathbf{F}}$.

Eq. (18): We need to show that $h(\mathsf{P}(\alpha \sqcup \beta) \leftrightarrow (\mathsf{P}\alpha \wedge \mathsf{P}\beta)) = 1_{\mathbf{F}}$. Then,

$$\begin{aligned} h(\mathsf{P}(\alpha \sqcup \beta) \leftrightarrow (\mathsf{P}\alpha \wedge \mathsf{P}\beta)) &= h((\neg\mathsf{P}(\alpha \sqcup \beta) \vee (\mathsf{P}\alpha \wedge \mathsf{P}\beta)) \\ &\qquad \wedge (\neg(\mathsf{P}\alpha \wedge \mathsf{P}\beta) \vee \mathsf{P}(\alpha \sqcup \beta))) \\ &= h(\neg\mathsf{P}(\alpha \sqcup \beta) \vee (\mathsf{P}\alpha \wedge \mathsf{P}\beta)) \\ &\qquad *_{\mathbf{F}}\, h(\neg(\mathsf{P}\alpha \wedge \mathsf{P}\beta) \vee \mathsf{P}(\alpha \sqcup \beta)) \end{aligned}$$

We continue by cases. Consider first:

$$\begin{aligned} h(\neg\mathsf{P}(\alpha \sqcup \beta) \vee (\mathsf{P}\alpha \wedge \mathsf{P}\beta)) &= h(\neg\mathsf{P}(\alpha \sqcup \beta)) +_{\mathbf{F}} h(\mathsf{P}\alpha \wedge \mathsf{P}\beta)) \\ &= -_{\mathbf{F}}\, h(\mathsf{P}(\alpha \sqcup \beta)) +_{\mathbf{F}} (h(\mathsf{P}\alpha) *_{\mathbf{F}} h(\mathsf{P}\beta)) \\ &= -_{\mathbf{F}}(h(\mathsf{P}\alpha) *_{\mathbf{F}} h(\mathsf{P}\beta)) +_{\mathbf{F}} (h(\mathsf{P}\alpha) *_{\mathbf{F}} h(\mathsf{P}\beta)) \\ &= 1_{\mathbf{F}} \end{aligned}$$

Similarly, $h(\neg(\mathsf{P}\alpha \wedge \mathsf{P}\beta) \vee \mathsf{P}(\alpha \sqcup \beta)) = 1_{\mathbf{F}}$.

Eq. (20) We need to show that $h((\mathsf{P}\alpha \wedge \mathsf{F}\alpha) \to (\alpha = \mathsf{0})) = 1_{\mathbf{F}}$. Then,

$$\begin{aligned} h((\mathsf{P}\alpha \wedge \mathsf{F}\alpha) \to (\alpha = \mathsf{0})) &= h(\neg(\mathsf{P}\alpha \wedge \mathsf{F}\alpha) \vee (\alpha = \mathsf{0})) \\ &= h(\neg(\mathsf{P}\alpha \wedge \mathsf{F}\alpha)) +_{\mathbf{F}} h(\alpha = \mathsf{0}) \\ &= -_{\mathbf{F}}\, h(\mathsf{P}\alpha \wedge \mathsf{F}\alpha) +_{\mathbf{F}} \mathscr{E}(h(\alpha), h(\mathsf{0})) \\ &= -_{\mathbf{F}}(\mathscr{P}(h(\alpha)) *_{\mathbf{F}} \mathscr{F}(h(\alpha))) +_{\mathbf{F}} \mathscr{E}(h(\alpha), 0_{\mathbf{A}}) \\ &= -_{\mathbf{F}}\, \mathscr{E}(h(\alpha), 0_{\mathbf{A}}) +_{\mathbf{F}} \mathscr{E}(h(\alpha), 0_{\mathbf{A}}) \\ &= 1_{\mathbf{F}} \end{aligned}$$

□

It is important to notice that, as expected, not every sentence is provable in DAL. In particular, if φ is a theorem, i.e., $\vdash \varphi$, then, $\neg\varphi$ is not provable, i.e., $\nvdash \neg\varphi$. This claim is substantiated as follows. Let $\mathbf{D} = \langle \mathbf{A}, \mathbf{F}, \mathscr{E}, \mathscr{P}, \mathscr{F} \rangle$ be the deontic action algebra in Fig. 2, and let h be any interpretation on $\mathbf{D}$; if φ is a theorem, then, $h(\varphi) = 1_{\mathbf{F}}$. Since h is a homomorphism, $h(\neg\varphi) = 0_{\mathbf{F}}$. Therefore, from Thm. 3.7, $\nvdash \neg\varphi$.

To prove the converse of Thm. 3.7, our sought after algebraic completeness result, we need to show that every non-theorem of DAL can be falsified on some deontic action algebra $\mathbf{D}$ (in the sense that there is some homorphism on $\mathbf{D}$ under which the non-theorem does not evaluate to $1_{\mathbf{F}}$). To this end, we introduce the notion of a Lindenbaum-Tarski deontic action algebra.

Fact 3.8 *Let* $\cong_a \subseteq \mathsf{Act} \times \mathsf{Act}$ *and* $\cong_f \subseteq \mathsf{Form} \times \mathsf{Form}$ *be defined as:*

$$\alpha \cong_a \beta \text{ iff} \vdash \alpha = \beta \qquad \varphi \cong_f \psi \text{ iff} \vdash \varphi \leftrightarrow \psi,$$

then the relations $\{\cong_a, \cong_f\}$ *form a congruence on the deontic action term algebra* $\mathbf{T}$*. This congruence is denoted with the symbol* $\cong$*.*

Definition 3.9 The Lindenbaum-Tarski deontic action algebra is the structure $\mathbf{L} = \langle \mathbf{A}, \mathbf{F}, \mathcal{E}, \mathcal{P}, \mathcal{F} \rangle$ where:

$$\mathbf{A} = \langle \mathsf{Act}/_{\cong_a}, \sqcup_{\cong_a}, \sqcap_{\cong_a}, {}^{-\cong_a}, [\mathsf{0}]_{\cong_a}, [\mathsf{1}]_{\cong_a} \rangle \qquad \mathcal{E}([\alpha]_{\cong_a}, [\beta]_{\cong_a}) = [\alpha = \beta]_{\cong_a}$$
$$\mathbf{F} = \langle \mathsf{Form}/_{\cong_f}, \vee_{\cong_f}, \wedge_{\cong_f}, \neg_{\cong_f}, [\bot]_{\cong_f}, [\top]_{\cong_f} \rangle \qquad \mathcal{P}([\alpha]_{\cong_a}) = [\mathsf{P}\alpha]_{\cong_f}$$
$$\mathcal{F}([\alpha]_{\cong_a}) = [\mathsf{F}\alpha]_{\cong_f}.$$

Proposition 3.10 *The Lindenbaum-Tarski deontic action algebra* $\mathbf{L}$ *is a deontic action algebra.*

Proof [Sketch] That $\mathbf{A}$ and $\mathbf{F}$ are Boolean algebras is more or less immediate. We show that the functions $\mathcal{E}$, $\mathcal{P}$, and $\mathcal{F}$ satisfy axioms from Eqs. (34), (35) and (39). The proof for axioms from Eqs. (36) to (38) are similar.

Eq. (34) We need to show that $\mathcal{P}([\alpha \sqcup \beta]_{\cong_a}) = \mathcal{P}([\alpha]_{\cong_a}) \wedge_{\cong_f} \mathcal{P}([\beta]_{\cong_a})$. Then,

$$\begin{aligned} \mathcal{P}([\alpha \sqcup \beta]_{\cong_a}) &= [\mathsf{P}(\alpha \sqcup \beta)]_{\cong_f} \\ &= [\mathsf{P}\alpha \wedge \mathsf{P}\beta]_{\cong_f} && \text{see Eq. (18)} \\ &= [\mathsf{P}\alpha]_{\cong_f} \wedge_{\cong_f} [\mathsf{P}\beta]_{\cong_f} \\ &= \mathcal{P}([\alpha]_{\cong_a}) \wedge_{\cong_f} \mathcal{P}([\beta]_{\cong_a}) \end{aligned}$$

Eq. (35) We need to show that $\mathcal{P}([\alpha]_{\cong_a}) \wedge_{\cong_f} \mathcal{F}([\alpha]_{\cong_a}) = \mathcal{E}([\alpha]_{\cong_a}, [\mathsf{0}]_{\cong_a})$. Then,

$$\begin{aligned} \mathcal{P}([\alpha]_{\cong_a}) \wedge_{\cong_f} \mathcal{F}([\alpha]_{\cong_a}) &= [\mathsf{P}\alpha]_{\cong_f} \wedge_{\cong_f} [\mathsf{F}(\alpha \sqcup \beta)]_{\cong_f} \\ &= [\mathsf{P}\alpha \wedge \mathsf{F}\alpha]_{\cong_f} \\ &= [\alpha = \mathsf{0}]_{\cong_f} && \text{see Eq. (20)} \\ &= \mathcal{E}([\alpha]_{\cong_a}, [\mathsf{0}]_{\cong_a}) \end{aligned}$$

Eq. (39) We need to show that $[\alpha]_{\cong_a} = [\beta]_{\cong_b}$ iff $\mathcal{E}([\alpha]_{\cong_a}, [\beta]_{\cong_a}) = [\top]_{\cong_f}$. Suppose that $[\alpha]_{\cong_a} = [\beta]_{\cong_b}$; it follows that $\vdash \alpha = \beta$; and so $\vdash (\alpha = \beta) \leftrightarrow \top$. Then, $\mathcal{E}([\alpha]_{\cong_a}, [\beta]_{\cong_a}) = [\alpha = \beta]_{\cong_f} = [\top]_{\cong_f}$. Similarly, if $\mathcal{E}([\alpha]_{\cong_a}, [\beta]_{\cong_a}) = [\top]_{\cong_f}$, then, $[\alpha]_{\cong_a} = [\beta]_{\cong_b}$.

□

The following result connects logical deduction in DAL with the Lindenbaum-Tarski Algebra. Roughly speaking, it says that the Lindenbaum-Tarski algebra captures DAL theoremhood.

Theorem 3.11 (Completeness) $\vdash \varphi$ *iff* $\mathbf{L} \models \varphi \approx \top$*.*

Proof The left to right direction is immediate from Thm. 3.7. For the right to left direction we show that if $\nvdash \varphi$, then $\mathbf{L} \not\models \varphi \approx \top$. Suppose that $\nvdash \varphi$; then $\nvdash \varphi \leftrightarrow \top$. This means that $[\varphi]_{\cong_f} \neq [\top]_{\cong_f}$. Construct an assignment $f : \mathsf{Act}_0 \to |\mathbf{L}|$ that sends each $\mathsf{a}_i \in \mathsf{Act}_0$ to the equivalence class $[\mathsf{a}_i]_{\cong_a}$. Using induction, we construct a homorphism f^* which agrees on f that is such that

$f^*(\varphi) = [\varphi]_{\cong_f}$. Then, from our assumption, we have $f^*(\varphi) = [\varphi]_{\cong_f} \neq [\top]_{\cong_f} = f^*(\top)$. Therefore, $\mathbf{L} \not\models \varphi \approx \top$. □

The following corollary can be obtained using a standard argument in algebraic logic.

Corollary 3.12 *If* $\models \varphi \approx \top$*, then* $\vdash \varphi$.

Proof *Assume* $\nvdash \varphi$*, then by Theorem 3.11, we have that* $\mathbf{L} \not\models \varphi \approx \top$ *and therefore* $\not\models \varphi \approx \top$. □

In other words, the Lindenbaum-Tsarski algebra can be thought as a canonical (algebraic) model which provides counterexamples of non-valid formulas.

3.3 Deontic Action Algebras and Deontic Action Models

We connect deontic action algebras and deontic action models via a Stone's representation. This gives us another proof of the completeness of Segerberg's deduction system w.r.t. the original semantics. Recall that the Stone's representation theorem [24] establishes that every Boolean algebra is isomorphic to a certain field of sets. We will prove a similar result for deontic action algebras.

We begin by introducing some additional concepts. First, just as Boolean algebras made of sets (i.e., fields of sets) are sometimes named *concrete Boolean algebras* in Algebraic Logic, we define concrete deontic action algebras as deontic action algebras whose action and formula algebras are fields of sets. Concrete deontic algebras allow us to establish the connection with Segerberg's original semantics for DAL.

Definition 3.13 A deontic action algebra $\mathbf{D} = \langle \mathbf{F}, \mathbf{A}, \mathcal{E}, \mathcal{P}, \mathcal{F} \rangle$ is called concrete iff $\mathbf{F}$ and $\mathbf{A}$ are fields of sets. The class of concrete deontic algebras is denoted by $\mathcal{C}_0$.

Using Stone duality we can prove that algebraic validity can be reduced to validity in concrete deontic algebras.

Theorem 3.14 *For any* DAL *formula* φ*, we have:* $\models \varphi \approx \top$ *iff* $\mathcal{C}_0 \models \varphi \approx \top$.

Proof The left to right direction is straightforward. For the other direction, assume that $\mathcal{C}_0 \models \varphi \approx \top$ and $\not\models \varphi \approx \top$. This means that we have a deontic action algebra $\mathbf{D} = \langle \mathbf{F}, \mathbf{A}, \mathcal{E}, \mathcal{P}, \mathcal{F} \rangle$ and a valuation v s.t. $\mathbf{D}, v \not\models \varphi \approx \top$. Applying Stone duality we have a concrete deontic action algebra $\mathbf{D}' = \langle \mathbf{F}', \mathbf{A}', \mathcal{E}', \mathcal{P}', \mathcal{F}' \rangle$ that is isomorphic to $\mathbf{D}$. On this concrete deontic algebra, we can define valuation $v'(a_i) = \varphi_{\mathbf{A}'}(v(a_i))$ (being $\varphi_{\mathbf{A}'}$ the Stone isomorphism for $\mathbf{A}'$). Then, we have $\mathbf{D}', v' \not\models \varphi \approx \top$. From this fact, we obtain a contradiction. □

We relate Segerberg's models to concrete deontic action algebras as follows.

Definition 3.15 Let $\mathfrak{M} = \langle E, P, F \rangle$ and $v : \mathsf{Act}_0 \to E$ be a deontic action model and a valuation, resp.; we associate with $\mathfrak{M}$ and v the deontic action algebra $\mathrm{alg}(\mathfrak{M}, v) = \langle \mathbf{F}^v_{\mathfrak{M}}, \mathbf{A}^v_{\mathfrak{M}}, \mathcal{E}^v_{\mathfrak{M}}, \mathcal{P}^v_{\mathfrak{M}}, \mathcal{F}^v_{\mathfrak{M}} \rangle$ where:

(a) $\mathbf{F}_{\mathfrak{M}} = \mathbf{2}$;

(b) $\mathbf{A}_{\mathfrak{M}}$ is the field of sets generated from $\{ v(\mathsf{a}_i) \mid \mathsf{a}_i \in \mathsf{Act}_0 \}$.

(c) $\mathcal{E}^v_{\mathfrak{M}}(x,y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$ (d) $\mathcal{P}^v_{\mathfrak{M}}(x) = \begin{cases} 1 & \text{if } x \subseteq P \\ 0 & \text{otherwise} \end{cases}$

(e) $\mathcal{F}^v_{\mathfrak{M}}(x) = \begin{cases} 1 & \text{if } x \subseteq F \\ 0 & \text{otherwise} \end{cases}$

Similarly, deontic action models form concrete deontic action algebras.

Definition 3.16 Let $\mathbf{D} = \langle \mathbf{F}, \mathbf{A}, \mathcal{E}, \mathcal{P}, \mathcal{F} \rangle$ be a concrete deontic action algebra and $f : \mathsf{Act}_0 \to A$ an assignment in $\mathbf{D}$; we associate with $\mathbf{D}$ and f a deontic action model $\mathrm{mod}(\mathbf{D}) = \langle E_{\mathbf{D}}, P_{\mathbf{D}}, F_{\mathbf{D}} \rangle$ and a valuation $v_f : \mathsf{Act}_0 \to E_{\mathbf{D}}$ where:

(a) $E_{\mathbf{D}} = |\mathbf{A}|$ (b) $P = \bigcup\{\, x \mid \mathbf{D}, f \approxeq \mathcal{P}(x) \approx \top \,\}$

(c) $F = \bigcup\{\, x \mid \mathbf{D}, f \approxeq \mathcal{F}(x) \approx \top \,\}$

The following are important properties of alg and mod.

Theorem 3.17 $\mathbf{D}, f \approxeq \varphi \approx \top$ *iff* $\mathrm{mod}(\mathbf{D}), v_f \Vdash \varphi$.

Theorem 3.18 $\mathfrak{M}, v \approxeq \varphi$ *iff* $\mathrm{alg}(\mathfrak{M}), f_v \approxeq \varphi \approx 1$.

Interestingly, when seen as operators, mod and alg are inverses of each other and therefore the two are isomorphisms.

Theorem 3.19 *For all deontic action algebra* $\mathbf{D}$ *and deontic action model* $\mathfrak{M}$*:*

$$\mathrm{alg}(\mathrm{mod}(\mathbf{D})) = \mathbf{D} \quad \textit{and} \quad \mathrm{mod}(\mathrm{alg}(\mathfrak{M})) = \mathfrak{M}$$

Then, we can prove the completeness of the Segerberg's deductive system w.r.t. deontic models in an algebraic way.

Theorem 3.20 $\vdash \varphi$ *iff* $\vDash \varphi$.

Proof Suppose $\vdash \varphi$. By algebraic completeness, this is equivalent to $\approxeq \varphi \approx \top$ and, by Thm. 3.14, also to $\mathcal{C}_0 \approxeq \varphi \approx \top$; by Thm. 3.19 and Thm. 3.17, this is equivalent to $\vDash \varphi$. □

4 Algebraizing Other Deontic Action Logics

The work of Segerberg in [23] gave rise to a family of closely related deontic logics. The logics DAL^i for $1 \leq i \leq 5$ reported in [26] are particularly interesting. Each DAL^i deals with a particular deontic issue, and is obtained from DAL^j (with $j < i$) by adding additional axioms to those in Fig. 1. Here, we show how to extend the algebraic framework in Sec. 3 to each of these variants.

The first of these extensions, DAL^1, is obtained from DAL by adding, for each $\mathsf{a}_i \in \mathsf{Act}_0$, $\mathsf{Fa}_i \vee \mathsf{Pa}_i$ to the set of axioms in Fig. 1. Intuitively, these axioms intend to capture what is called the *Principle of Deontic Closure* in deontic logics: what is not forbidden is permitted (alt., every action is either permitted or forbidden). As noted in [26], these axioms capture closeness only at the level of action generators, and they are not able to capture closeness for other (perhaps more fine-grained) actions. The algebraic counterpart of DAL^1 is determined by the class of deontic action algebras: (i) whose algebra of actions is freely generated by a set G of generators; and (ii) that satisfy Eq. (42) below.

$$\mathcal{F}(x) +_{\mathbf{F}} \mathcal{P}(x) = 1_{\mathbf{F}} \qquad \text{for every generator } x \in |\mathbf{A}| \tag{42}$$

In turn, the extension DAL^2 is obtained from DAL^1 by: (i) requiring the set Act_0 of basic actions to be a finite, i.e., $\mathsf{Act}_0 = \{\, \mathsf{a}_i \mid 0 \leq i \leq n \,\}$; and (ii) adding the axioms $\mathsf{P}(\overline{\mathsf{a}_0} \sqcap \cdots \sqcap \overline{\mathsf{a}_n}) \vee \mathsf{F}(\overline{\mathsf{a}_0} \sqcap \cdots \sqcap \overline{\mathsf{a}_n})$. Intuitively, the additional axiom states that not performing any of the basic actions is permitted or forbidden. On the algebraic side, by considering a finite set Act_0 of basic actions, we obtain that the algebra $\mathbf{A}$ of actions is an atomic Boolean algebra. The atoms in this algebra allow us to focus on the most basic actions being considered. Then, the algebraic counterpart of DAL^2 is determined by the class of deontic action algebras: (i) that are finitely generated by a set $G = \{\, a_i \mid 0 \leq i \leq n \,\}$; and that satisfy Eq. (43) below.

$$\mathcal{P}((-_{\mathbf{A}}\, a_1) +_{\mathbf{A}} \ldots +_{\mathbf{A}} (-_{\mathbf{A}}\, a_n)) +_{\mathbf{F}} \mathcal{F}((-_{\mathbf{A}}\, a_1) +_{\mathbf{A}} \ldots +_{\mathbf{A}} (-_{\mathbf{A}}\, a_n)) = 1_{\mathbf{F}} \tag{43}$$

The extension DAL^3 is obtained from DAL^2 by adding the following axiom: $(\mathsf{a}_1 \sqcup \cdots \sqcup \mathsf{a}_n) = 1_{\mathbf{A}}$. Intuitively, this axiom can be read as stating that the actions $\mathsf{a}_1, \ldots, \mathsf{a}_n$ are the sole actions that the agent can perform. The algebraic counterpart of DAL^3 is determined by the subclass of deontic action algebras of DAL^2 that further satisfy Eq. (44) below.

$$a_0 +_A \cdots +_A a_n = 1_A \tag{44}$$

The extension DAL^4 is obtained from DAL by requiring closedness at the level of "atomic" actions. Formally, DAL^4 considers a finite number of actions $\mathsf{Act}_0 = \{\, \mathsf{a}_i \mid 0 \leq i \leq n \,\}$ and a collection $\{\, \alpha_i \mid 0 \leq i \leq 2^n \,\}$ of action terms s.t.: each α_i is of the form $*\mathsf{a}_0 \sqcap \cdots \sqcap *\mathsf{a}_n$, where $*\mathsf{a}_i \in \{\mathsf{a}_i, \overline{\mathsf{a}_i}\}$. Syntactically, each α_i represents a possible atomic action. Closedness is then obtained by adding the following set of axioms to those of DAL: $\mathsf{P}\alpha_i \vee \mathsf{F}\alpha_i$, for all $0 \leq i \leq 2^n$. The algebraic counterpart of DAL^4 is determined by the class of deontic action algebras: (i) that are finitely generated; and (ii) that satisfy Eq. (45) below.

$$\mathcal{P}(a) +_{\mathbf{F}} \mathcal{F}(a) = 1_{\mathbf{F}} \qquad \text{for all atoms } a \in |\mathbf{A}| \tag{45}$$

Finally, the extension DAL^5 is obtained by putting together DAL^3 and DAL^4. The algebraic counterpart of DAL^5 is obtained from the deontic action algebras that are deontic action algebras of DAL^3 and DAL^4.

Following from the above, we obtain for each DAL^i an associated class $\mathcal{D}_i$ of deontic action algebras. Each of these classes accommodates for a corresponding soundness and completeness result. This is made precise in Thm. 4.1. (The proof of Thm. 4.1 is a routine extension of the proof of Thm. 3.11.)

Theorem 4.1 *For every $0 \leq i \leq 5$, let $\vdash_{\mathsf{DAL}^i}$ be theoremhood relation of DAL^i and $\vDash_{\mathcal{D}_i}$ equational validity in the class $\mathcal{D}_i$; then, $\vdash_{\mathsf{DAL}^i} \varphi$ iff $\vDash_{\mathcal{D}_i} \varphi \approx \top$.*

5 Final Remarks

We presented an algebraic treatment of Sergerberg's deontic action logic and some of known extensions via deontic action algebras. As is commonly done in the algebraization of a logic, along the way we discussed concepts such as: actions and formulas algebras, operators of permission and prohibition, and Lindenbaum-Tarski algebras. Moreover, we established that the algebraic characterization is correct by proving soundness and completeness theorems. In our opinion the overall picture is just as important. Our algebraic treatment can be thought of as an abstract version of deontic action logics which can be used to establish connections between deontic action logics and mathematical areas such as topology, category theory, probability, etc.

In addition to the obvious mathematical benefits of having an algebraization of deontic action logics, we believe that the algebraic framework introduced above paves the way for interesting future work. First, deontic action algebras are modular in their formulation; i.e., action and formula algebras can be replaced to obtain new systems. For instance, by changing the algebra of actions we can obtain systems where it is possible to reason about other action combinators. Interesting cases are those of: action composition (denoted by ;), and action iteration (denoted by $*$). In this line, the work of Meyer in [12] was one of the first in considering a deontic logic containing action composition. Meyer named the system Dynamic Deontic Logic (DDL). This system is not without challenges. As observed in [3], one of the main problems of DDL is that action composition (and so action iteration) makes it posible to formulate some paradoxes. Regarding action iteration ($*$), in [8], Broersen pointed out that dynamic deontic logics can be divided into: (i) *goal norms*, where prescriptions over a sequence of actions only take into account the last action performed; or (ii) *process norms*, where a sequence of actions is permitted/forbidden iff every action in the sequence is permittedd/forbidden. It is a matter of discussion which one of these approaches is better, but both have cons and pros. The interested reader is referred to [8] for an in-depth discussion on this issue. To the best of our knowledge, we are not aware of any extension of Segerberg's logic that provides action composition or action iteration. This said, notice that deontic action algebras can be straightforwardly modified to admit these operators. More precisely, we may consider deontic action algebras $\langle \mathbf{F}, \mathbf{A}, \mathscr{P}, \mathscr{F} \rangle$ where $\mathbf{F}$ is a Boolean algebra; $\mathbf{A} = \langle A, +, ;, * \rangle$ is a Kleene algebra (see [15]); and $\mathscr{P}$ and $\mathscr{F}$ are deontic operators of permitted and forbidden on these algebras. Intuitively, in $\mathbf{A}$, ; captures action composition, $+$ captures action choice, and $*$ captures the iteration of actions. Kleene algebras enjoy some nice properties. They are quasi-varieties, and they are complete w.r.t. equality of regular expressions (see [16]). In this respect, Kleene algebras provide a robust framework for reasoning about action composition and iteration. Similarly, one can extend deontic action algebras with other interesting algebras; e.g., relation algebras (see [17]). Relation algebras would provide other action operators, most notably, action converse. We leave it as further work investigating the properties the operators $\mathscr{P}$ and $\mathscr{F}$ in these new algebraic settings.

In turn, another interesting line of research consists in investigating other algebras for formulas. In this paper, we have used Boolean algebras as an abstraction of formulas, but there are different kinds of algebras that may provide alternative ways for reasoning about norms. Some immediate examples are: Heyting Algebras, semi-lattices, metric spaces, etc. We draw attention to the fact that changing the algebra of formulas in deontic action algebras may bridge the way for designing deontic logics that are not logics of normative propositions. More precisely, von Wright in [29], and Alchourron in [1,2], both noted the distinction between logics of normative propositions and logics of norms. The former are Boolean logics where their formulas express assertions about the existence of norms; i.e., a formula s.t. $\mathsf{P}\varphi$ states that *there is a norm allowing the occurrence of* φ – SDL and DAL fall into this category. In contrast, logics of norms allow to express prescriptions that, as observed by von Wright, are not necessarily evaluated to a Boolean value (i.e., true or false). To deal with logics of norms, we can use other algebras to generalize formulas. For instance, by taking a meet semi-lattice as the algebra of formulas we can capture a theory of norms where norms can be put together, and where some norms are in contradiction with each other (but not necessarily where norms are true or false). Of course, there are several other appealing algebras that could play this role as well: metric spaces, rings, etc. We leave all this as a further work.

Appendix

A Many Sorted Algebras in a Nutshell

In this section we introduce some basic concepts used in the paper. These serve as a way to fix terminology and notation. The interested reader is referred to [10,22] for an in-depth introduction to this topic.

Definition A.1 A many-sorted signature is a pair $\Sigma = \langle S, \Omega \rangle$ where: (a) S is a set of sort names; and (b) $\Omega = \{ f : s_1 \dots s_n \to s \mid s_i, s \in S \}$ is a set of operation names. A Σ-algebra $\mathbf{A}$ consists of: (c) an S-indexed family of sets, written $|\mathbf{A}| = \{ A_s \mid s \in S \}$; and (d) for each $f : s_1 \dots s_n \to s \in \Omega$ a function $f_{\mathbf{A}} : A_{s_1} \dots A_{s_n} \to A_s$.

Note that standard algebras can be seen as many-sorted algebras with only one sort. A special kind of Σ-algebras are the so-called Σ-term algebras.

Definition A.2 Let $\Sigma = \langle S, \Omega \rangle$ be a signature and $X = \{ X_s \mid s \in S \}$ be an S-indexed family of sets; a Σ-term algebra with variables in X is a Σ-algebra $\mathbf{T}$ in which:

(a) $|\mathbf{T}| = \{ T_s \mid s \in S \}$ is the $\subseteq$-smallest S-indexed family of sets s.t. for all $x \in X_s$, the string 'x' $\in T_s$; and for all $f : s_1 \dots s_n \to s \in \Omega$ and strings $t_i \in T_{s_i}$, the string '$f(t_1 \dots t_n)$' $\in T_s$;

(b) for each $f : s_1 \dots s_n \to s \in \Omega$, there is a function $f_{\mathbf{T}} : T_{s_1} \dots T_{s_n} \to T_s$ s.t. for all strings $t_i \in T_{s_i}$, $f_{\mathbf{T}(X)}(t_1 \dots t_n)$ equals the string '$f(t_1 \dots t_n)$'.

Definition A.3 Let $\mathbf{A}$ and $\mathbf{B}$ be Σ-algebras; a Σ-homomorphism $h : \mathbf{A} \to \mathbf{B}$ is an S-indexed family of functions $h = \{ h_s : A_s \to B_s \mid s \in S \}$ such that: for all $f : s_1 \dots s_n \to s \in \Omega$ and $a_i \in A_{s_i}$, it follows that $h_s(f_{\mathbf{A}}(a_1 \dots a_n)) = f_{\mathbf{B}}(h_{s_1}(a_1) \dots h_{s_n}(a_n))$.

Definition A.4 Let $\mathbf{A}$ be a Σ-algebra; a Σ-congruence $\cong$ on $\mathbf{A}$ is an S-sorted family of relations $\cong\, = \{ \cong_s \subseteq {A_s}^2 \mid s \in S \}$ such that: each $\cong_s$ is an equivalence relation on A_s^2; and for all $f : s_1 \ldots s_n \to s \in \Omega$ and $a_i, a'_i \in A_{s_i}$, if $a_i \cong_{s_i} a'_i$, then $f_{\mathbf{A}}(a_1 \ldots a_n) \cong_s f_{\mathbf{A}}(a'_1 \ldots a'_n)$.

Definition A.5 Let $\mathbf{A}$ be a Σ-algebra and $\cong$ be a Σ-congruence on $\mathbf{A}$; the quotient Σ-algebra of $\mathbf{A}$ under $\cong$, written $\mathbf{A}/_{\cong}$, has: (a) $|\mathbf{A}/_{\cong}| = \{ A_s/_{\cong_s} \mid s \in S \}$; and (b) for all $f : s_1 \ldots s_n \to s \in \Omega$ and $a_i \in A_{s_i}$, $f_{\mathbf{A}/_{\cong}}([a_1]_{\cong_{s_1}} \ldots [a_n]_{\cong_{s_n}}) = [f_{\mathbf{A}}(a_1 \ldots a_n)]_{\cong_s}$.

We omit making sorts and indices from signatures explicit when they can easily be understood from the context. We also omit making an explicit distinction between signatures and algebras. Moreover, making an abuse of notation, we indicate a Σ-algebras by its signature Σ. By this, we mean a Σ-algebra which has no other function than those named in Σ. We conclude this section by recalling some basics definitions of Boolean algebras.

Definition A.6 A Boolean algebra is an algebra $\mathbf{A} = \langle A, +, *, -, 0, 1 \rangle$ where: (a) $A = |\mathbf{A}|$ is a non-empty set of elements; and (b) $+, * : A^2 \to A$ are commutative and associative; $- : A \to A$ is idempotent; and $0, 1 : A$, called top and bottom, are neutral elements for $+$ and $*$, resp., further satisfying for all $a \in A$, $a + -a = 1$ and $a * -a = 0$.

Definition A.7 Every Boolean algebra $\mathbf{A}$ is equipped with a partial order defined as $x \preceq_{\mathbf{A}} y$ iff $x = x * y$. An ideal is a non-empty subset $I \subseteq |\mathbf{A}|$ s.t.: (a) for all $x, y \in I$, there is $z \in I$ s.t. $z \preceq_{\mathbf{A}} x * y$; and (b) for all $x \in I$ and $a \in |\mathbf{A}|$, if $a \preceq_{\mathbf{A}} x$, then, $a \in I$. An ideal I is proper if $I \neq |\mathbf{A}|$; otherwise it is trivial. An ideal I is maximal if there is no other ideal J s.t. $I \subset J$. The smallest ideal containing an element $a \in |\mathbf{A}|$, called a principal ideal, is the set $\downarrow a = \{ x \mid x \preceq_{\mathbf{A}} a \}$. The dual notion of an ideal is called a filter and is obtained by reversing $\preccurlyeq_{\mathbf{A}}$ and exchanging $*$ with $+$.

Two other notions that are important in our constructions are: freely generated and finitely generated algebras.

Definition A.8 Let $\mathbf{A} = \langle A, +, *, -, 0, 1 \rangle$ be a Boolean algebra; a subset $E \subseteq A$ is called a set of generators for $\mathbf{A}$ iff the following facts hold: (a) the intersection of all subalgebras of $\mathbf{A}$ including E is a subalgebra; (b) that intersection is the smallest subalgebra of $\mathbf{A}$ including E. Such algebra is called the generated algebra. It is called finitely generated, if the set of generators E is finite.

Definition A.9 A set E of generators of a Boolean algebra $\mathbf{B}$ is called free if every mapping from E to an arbitrary Boolean algebra $\mathbf{A}$ can be uniquely extended to an homomorphism $h : \mathbf{B} \to \mathbf{A}$. An algebra is called freely generated (or free) if it has a free set of generators.

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