Default Modal Systems as Algebraic Updates

Valentin Cassano[1,2,3], Raul Fervari[1,2], Carlos Areces[1,2], and Pablo F. Castro[1,3]

[1] Consejo Nacional de Investigaciones Científicas y Técnicas, Argentina
[2] Universidad Nacional de Córdoba, Argentina
[3] Universidad Nacional de Río Cuarto, Argentina

Abstract. Default Logic refers to a family of formalisms designed to carry out non-monotonic reasoning over a monotonic logic (in general, Classical First-Order or Propositional Logic). Traditionally, default logics have been defined and dealt with via syntactic consequence relations. Here, we introduce a family of default logics defined over modal logics. First, we present these default logics syntactically. Then, we elaborate on an algebraic counterpart. We do the latter by extending the notion of a modal algebra to acommodate for the main elements of default logics: defaults and extensions. Our algebraic treatment of default logics concludes with an algebraic completeness result. To our knowledge, our approach is novel, and it lays the groundwork for studying default logics from a dynamic logic perspective.

1 Introduction

Default Logic refers to a family of non-monotonic formalisms tailored to reasoning with incomplete knowledge, and to dealing with contradictory information. The main features of a default logic DL are defaults and extensions. Intuitively, defaults can be seen as defeasible rules of inference, i.e., rules of inference whose conclusions are subject to annulment. Defaults are used as a tool to handle reasoning from incomplete knowledge. In turn, extensions can be understood as sets of formulas closed under the application of defaults. Extensions are a way of reasoning in the presence of contradictory information (via consistent alternatives).

The history of Default Logic traces back to Reiter's seminal work [20]. Since then, many variants of Reiter's original ideas have been proposed – each variant has given rise to a different default logic (see [2] for a comprehensive summary). For the most part, these variants have focused their attention on what is meant by an extension. In particular, the emphasis has been on how different interactions between defaults, and the rules of inference of the underlying proof calculus,[4] concoct different notions of an extension satisfying one or more properties of interest. This treatment of extensions carries with it the definition and analysis of a default logic from a syntactic perspective. At the same time, in studying a logic (of any kind), we also wish to address it from a semantic perspective via

[4] Typically the underlying proof calculi is one for Classical First-Order Logic (FOL) (see, e.g., [20]) or for Classical Propositional Logic (CPL) (see, e.g., [16,21,5,18]).

a model theory and/or a class of algebras. This yields interesting completeness results, interpolation properties, bisimulations, etc. This semantic perspective on default logics is mostly absent, making it difficult to investigate their logical properties using standard semantic tools.

Our work. Following the tradition in Default Logic, we start with a formulation of default logics over modal logics via deducibility (i.e., syntactical consequence in the proof calculus). We rely on the notion of global deducibility for modal logics [9]. Our formulation of a default logic is parametric, and can be instantiated with any modal system from K to $\mathsf{S5}$ extended with the universal modality [3].

For each default modal logic, we make explicit how defaults interact with the rules of inference of the underlying proof calculus. To this end, we integrate the use of defaults into the notion of deducibility in the underlying proof calculus. In addition, we show how we can parametrically define for each default modal system an algebraic counterpart. We do this by extending modal algebras to accommodate for defaults and extensions. Modal algebras are Boolean algebras with additional operators for modalities, and they make up the algebraic counterpart of modal systems [27,11].

The algebraic treatment of defaults and extensions is done as follows. We carry out a Lindenbaum-Tarski construction that acts as an algebraic canonical model for a set of permisses. We enrich this construction with an operator to deal with defaults. This operator can be thought of as "updating" the Lindenbaum-Tarski algebra w.r.t. the application of a default. The result of the update is the algebraic counterpart of an extension. On this basis we prove an algebraic completeness result.

Related work. Our treatment of defaults and extensions enables us to think of default logics as algebraic model changing logics; in the sense of, e.g., public announcement logic [19].

In our case, a model update corresponds to the application of a default (a sort of inference step). The idea of updating a model dynamically to represent syntactic steps of inference can be found in several places in the literature on dynamic logics. For instance, the problem of logical omniscience in epistemic logic (see, e.g., [25]) has been thought of as a property to be achieved after the application of a dynamic operation. In [6,1,15,22], omniscience is achieved by updating models containing sets of formulas. In [24,14] the updates are performed over awareness relational models. Dynamics of evidence are presented in [23,26] over neighbourhood models. Finally, dynamic modalities allowing to achieve introspective states over Kripke models are introduced in [7,8].

Closer to our work is the algebraic treatment of public announcements introduced in [17]. Therein, the algebraic submodel relation induced by the announcement of a formula ψ is represented by taking the quotient algebra modulo an equivalence relation given by ψ. We show that the application of a default δ can be captured in a similar way, i.e., by taking the quotient algebra modulo the equivalence relation given by the conclusion of δ.

Motivation. Our choice of defining default logics over modal logics is not arbitrary. Modal logics provide a wide spectrum of logics which are more expressive than CPL, with better computational properties than FOL. Moreover, these logics have a well-developed algebraic theory in terms of modal algebras. In our constructions we exploit the combination of these two features. As we will see, defaults are better modelled by means of a global consequence relation, which will be captured by the use of the universal modality.

Main contributions. We provide a syntactic and algebraic treatment of default logics built over modal logics and study their properties. Syntactically, our construction of a default modal system is parametric on a modal system and a set of defaults. We make precise how defaults interact with the rules of inference of the underlying modal system. Algebraically, we address defaults and extensions via modal algebras. This enables us to obtain an algebraic completeness result. Moreover, it enables us the use of standard algebraic tools to study metalogical properties of default modal systems.

We view this work as a first step towards an algebraization of default logic, and towards a better understanding of default systems from a logical perspective. Finally, the algebraic construction for default logics over modal logics, enables us to study default systems from a dynamic logic perspective.

Structure of the article. Sec. 2 covers background material. Sec. 3 contains our main results. Sec. 3.1 introduces default modal systems. Sec. 3.2 presents default deducibility. Sec. 4 provides our algebraic characterization of defaults and extensions, and a completeness theorem. In Sec. 4 we discuss default modal systems from a dynamic logic perspective. In Sec. 5 we offer some final remarks.

2 Background

2.1 Boolean Algebra in a Nutshell

We introduce some definitions and notation for Boolean algebras (see, e.g., [12] for details).

Definition 1. *A* Boolean Algebra *(BA) is a structure* $\mathbf{A} = \langle A, *, -, 1\rangle$ *satisfying a well-known set of equations. A is also denoted as* $|\mathbf{A}|$*. Occasionally, we consider operations* $+$ *and* 0 *defined as* $a + b = -(-a * -b)$*, and* $0 = -1$*.*

Definition 2. *Every BA* $\mathbf{A}$ *brings in a partial order* $\preceq_{\mathbf{A}}$ *defined as* $x \preceq_{\mathbf{A}} y$ *iff* $x = x * y$ *(sometimes we omit the subindex* $\mathbf{A}$ *and write just* $\preceq$*). We write* $\uparrow X = \{ y \mid \text{there is } x \in X \text{ s.t. } x \preceq y \}$*. A filter is a non-empty subset* $F \subseteq |\mathbf{A}|$ *s.t.:* $F = \uparrow F$ *and for all* $x, y \in F$*,* $(x * y) \in F$*. A filter is principal if it is of the form* $\uparrow\{a\}$ *for* $a \in |\mathbf{A}|$*. A filter* F *is* proper *if* $0 \notin F$*.*

2.2 Modal Systems

We make precise the notion of a modal system over the set Form of wffs below.

Definition 3. *Let* $\mathsf{Prop} = \{\, p_i \mid i \in \mathbb{N} \,\}$ *be a denumerable set of* proposition symbols*; the set* Form *of wffs is determined by the grammar*

$$\varphi, \psi ::= p_i \mid \top \mid \neg\varphi \mid \varphi \wedge \psi \mid \Box\varphi \mid [\mathsf{u}]\varphi.$$

We use $\bot$*,* $\varphi \vee \psi$*,* $\varphi \to \psi$*,* $\varphi \leftrightarrow \psi$*,* $\Diamond\varphi$ *and* $\langle\mathsf{u}\rangle\varphi$ *as abbreviations defined in the usual way.*

The set Form of wffs can be seen as an enrichment of the basic modal language with the universal modality $[\mathsf{u}]$. We use the universal modality as a technical tool to internalize a global consequence relation.

A modal system is determined by a subset of Form, called axioms, and the rules of inference in Def. 4.

Definition 4. *The set of* rules of inference *of a modal system consists of*

$$\frac{\varphi \quad \varphi \to \psi}{\psi}\ (\mathsf{mp}) \qquad\qquad \frac{\varphi}{[\mathsf{u}]\varphi}\ (\mathsf{u}).$$

The modal system $\mathsf{K}^{[\mathsf{u}]}$ is determined by the axioms in Def. 5.

Definition 5. *The* axioms *of* $\mathsf{K}^{[\mathsf{u}]}$ *is the smallest set of formulas which contains all instances of* propositional tautologies *and the* schemas*:*

1. $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$*;*
2. $[\mathsf{u}](\varphi \to \psi) \to ([\mathsf{u}]\varphi \to [\mathsf{u}]\psi)$*;*
3. $[\mathsf{u}]\varphi \to \varphi$*;*
4. $\varphi \to [\mathsf{u}]\langle\mathsf{u}\rangle\varphi$*;*
5. $[\mathsf{u}]\varphi \to [\mathsf{u}][\mathsf{u}]\varphi$*;*
6. $[\mathsf{u}]\varphi \to \Box\varphi$*.*

We take $\mathsf{K}^{[\mathsf{u}]}$ as our basic modal system. The rest of the modal systems we consider are constructed by enlarging the set of axioms of $\mathsf{K}^{[\mathsf{u}]}$ with (all instances of) any of the schemas below, or any combination thereof, as additional axioms.

(4) $\Box\varphi \to \Box\Box\varphi$ (5) $\Diamond\varphi \to \Box\Diamond\varphi$ (B) $\varphi \to \Box\Diamond\varphi$ (D) $\Box\varphi \to \Diamond\varphi$ (T) $\Box\varphi \to \varphi$

E.g., the system $\mathsf{D}^{[\mathsf{u}]}$ is obtained by adding to the axioms of $\mathsf{K}^{[\mathsf{u}]}$ all instances of the schema D as further axioms. Similarly, the systems $\mathsf{S4}^{[\mathsf{u}]}$ and $\mathsf{S5}^{[\mathsf{u}]}$ are obtained by adding the schemas T and 4, and T and 5, respectively.

For each modal system M, we define a consequence relation $\vdash_{\mathsf{M}}$ between sets of formulas and formulas. This relation is made precise in Def. 6.

Definition 6. *Let* M *be a modal system; an* M-deduction of φ from Φ *is a finite sequence* $\psi_1 \ldots \psi_n$ *of formulas of* Form *such that* $\psi_n = \varphi$*, and for each* $k < n$*:*

1. ψ_k *is an axiom of* M*;*
2. ψ_k *is a premiss, i.e.,* $\psi_k \in \Phi$*;*
3. ψ_k *is obtained from two earlier formulas using* mp*, i.e., there are* $i, j < k$ *s.t.* $\psi_j = \psi_i \to \psi_k$*;*
4. ψ_k *is obtained from an earlier formula using* u*, i.e., there is* $j < k$ *s.t.* $\psi_k = [\mathsf{u}]\psi_j$*.*

We write $\Phi \vdash_{\mathsf{M}} \varphi$ iff there is an M*-deduction of φ from Φ. The relation $\vdash_{\mathsf{M}}$ is commonly referred to as* global consequence.

If there is no need to distinguish between modal systems, we simply speak of a relation $\vdash$ and of a deduction.

We end this section by taking note of the following properties of $\vdash_{\mathsf{M}}$. Notice that the first item refers to the *necessitation* property in modal logics, whereas the second item refers to a version of the *deduction theorem*.

Proposition 1. *The following properties hold:*

1. *If $\vdash_{\mathsf{M}} \varphi$, then, $\vdash_{\mathsf{M}} \Box\varphi$.*
2. *If $\Phi \cup \{\varphi\} \vdash_{\mathsf{M}} \psi$, then, $\Phi \vdash_{\mathsf{M}} \boxdot\varphi \to \psi$.*

2.3 Algebraizing Modal Systems

We present the semantics of a modal system from an algebraic perspective. Following [27], and borrowing ideas and results from [11,13], we associate with any modal system M a suitable class of algebras in a way such that the properties of M are in correspondence to the properties of this class.

For the case of the modal systems we consider we will use $\boxdot$-modal algebras. We use this algebraic treatment of modal systems to perform default reasoning from a semantic point of view. This algebraic treatment is also instrumental to viewing default reasoning as a logic of *updates* over algebras. But this is us getting ahead of ourselves. For now, we focus on introducing some basic concepts and results regarding $\boxdot$-modal algebras.

Definition 7. *The* formula algebra *corresponding to the set* Form *of formulas is the structure $\mathbf{F} = \langle \mathsf{Form}, \wedge, \neg, \top, \Box, \boxdot \rangle$ where: $\neg$, $\Box$, $\boxdot$ are unary functions on* Form*, and $\wedge$ is a binary function on* Form*, such that $\neg$ applied to $\varphi \in \mathsf{Form}$ returns $\neg\varphi \in \mathsf{Form}$, $\Box$ applied to $\varphi \in \mathsf{Form}$ returns $\Box\varphi \in \mathsf{Form}$, $\boxdot$ applied to $\varphi \in \mathsf{Form}$ returns $\boxdot\varphi \in \mathsf{Form}$, and $\wedge$ applied to $\varphi, \psi \in \mathsf{Form}$ returns $\varphi \wedge \psi \in \mathsf{Form}$.*

Just as Boolean algebras (as interpretation structures) and filters (as the semantic counterpart of deducibility) are fundamental for the algebraization of Classical Propositional Logic, $\boxdot$-modal algebras and *open filters* are fundamental for the algebraization of modal systems.

Definition 8. *A* $\boxdot$-modal algebra *is a structure $\mathbf{M} = \langle B, *, -, 1, f^{\Box}, f^{\boxdot} \rangle$ where: $\langle B, *, -, 1 \rangle$ is a Boolean algebra; and $f^{\Box}$ and $f^{\boxdot}$ are unary functions on B satisfying the following equations*

$$
\begin{array}{rlrl}
f^{\Box}(1) &= 1 & f^{\boxdot}(b_1) &\preccurlyeq b_1 \\
f^{\Box}(b_1 * b_2) &= f^{\Box}(b_1) * f^{\Box}(b_2) & f^{\boxdot}(b_1) &\preccurlyeq f^{\boxdot}(-f^{\boxdot}(-b_1)) \\
f^{\boxdot}(1) &= 1 & f^{\boxdot}(b_1) &\preccurlyeq f^{\boxdot} f^{\boxdot}(b_1) \\
f^{\boxdot}(b_1 * b_2) &= f^{\boxdot}(b_1) * f^{\boxdot}(b_2) & f^{\boxdot}(b_1) &\preccurlyeq f^{\Box}(b_1).
\end{array}
$$

An open filter is a subset $F \subseteq B$ such that F is a filter in $\langle B, *, -, 1 \rangle$, and for all $b \in F$, $f^{\boxdot}(b) \in F$.

Definition 9. *An* interpretation *of the formula algebra* $\mathbf{F}$ *on a* $\boxdot$*-modal algebra* $\mathbf{M} = \langle B, *, -, 0, f^{\Box}, f^{\boxdot}\rangle$*, a.k.a. an interpretation on* $\mathbf{M}$*, is a homomorphism* $v : \mathbf{F} \to \mathbf{M}$ *such that:*

$$v(\top) = 1 \qquad v(\neg\varphi) = -v(\varphi) \qquad v(\Box\varphi) = f^{\Box}(v(\varphi))$$
$$v(\varphi \land \psi) = v(\varphi) * v(\psi) \qquad v(\boxdot\varphi) = f^{\boxdot}(v(\varphi)).$$

Proposition 2. *Every interpretation* v *on* $\mathbf{M}$ *is uniquely determined by an assignment* $v_0 : \mathsf{Prop} \to |\mathbf{M}|$.

Definition 10. *Let* $\mathbf{M}$ *be a* $\boxdot$*-modal algebra; we define:*

1. *an* equation *is a member of* Form^2*; we write an equation* (φ, ψ) *as* $\varphi \approx \psi$*;*
2. *an equation* $\varphi \approx \psi$ *is* valid under an interpretation v on $\mathbf{M}$ *iff* $v(\varphi) = v(\psi)$*; we write* $\mathbf{M}, v \vDash \varphi \approx \psi$ *if* $\varphi \approx \psi$ *is valid under* v*;*
3. *an equation* $\varphi \approx \psi$ *is* valid in $\mathbf{M}$ *iff* $\mathbf{M}, v \vDash \varphi \approx \psi$ *for all interpretations* v *on* $\mathbf{M}$*; we write* $\mathbf{M} \vDash \varphi \approx \psi$ *if* $\varphi \approx \psi$ *is valid in* v.

We are now in a position to connect $\boxdot$-modal algebras and modal systems.

Proposition 3. *Let* M *be a modal system; the relation* $\cong^{\Phi}_{\mathsf{M}}$ *defined as:* $\varphi \cong^{\Phi}_{\mathsf{M}} \psi$ *iff* $\Phi \vdash_{\mathsf{M}} \varphi \leftrightarrow \psi$ *yields a congruence on* $\mathbf{F}$.

Definition 11. *Let* M *be a modal system; the* M-Lindenbaum-Tarski algebra *of a set* Φ *of wffs is the structure* $\mathbf{L}^{\Phi}_{\mathsf{M}} = \langle \mathsf{Form}/_{\cong^{\Phi}_{\mathsf{M}}}, *_{\cong^{\Phi}_{\mathsf{M}}}, -_{\cong^{\Phi}_{\mathsf{M}}}, 1_{\cong^{\Phi}_{\mathsf{M}}}, f^{\Box}_{\cong^{\Phi}_{\mathsf{M}}}, f^{\boxdot}_{\cong^{\Phi}_{\mathsf{M}}}\rangle$ *where:* $\mathsf{Form}/_{\cong^{\Phi}_{\mathsf{M}}} = \{\, [\varphi]_{\cong^{\Phi}_{\mathsf{M}}} \mid \varphi \in \mathsf{Form}\,\}$*; and*

$$1_{\cong^{\Phi}_{\mathsf{M}}} = [\top]_{\cong^{\Phi}_{\mathsf{M}}} \qquad -_{\cong^{\Phi}_{\mathsf{M}}}([\varphi]_{\cong^{\Phi}_{\mathsf{M}}}) = [\neg\varphi]_{\cong^{\Phi}_{\mathsf{M}}} \qquad f^{\Box}_{\cong^{\Phi}_{\mathsf{M}}}([\varphi]_{\cong^{\Phi}_{\mathsf{M}}}) = [\Box\varphi]_{\cong^{\Phi}_{\mathsf{M}}}$$
$$[\varphi]_{\cong^{\Phi}_{\mathsf{M}}} *_{\cong^{\Phi}_{\mathsf{M}}} [\psi]_{\cong^{\Phi}_{\mathsf{M}}} = [\varphi \land \psi]_{\cong^{\Phi}_{\mathsf{M}}} \qquad f^{\boxdot}_{\cong^{\Phi}_{\mathsf{M}}}([\varphi]_{\cong^{\Phi}_{\mathsf{M}}}) = [\boxdot\varphi]_{\cong^{\Phi}_{\mathsf{M}}}.$$

The canonical interpretation v *on* $\mathbf{L}^{\Phi}_{\mathsf{M}}$ *is defined as* $v(\varphi) = [\varphi]_{\cong^{\Phi}_{\mathsf{M}}}$.

Proposition 4. *Every* M*-Lindenbaum-Tarski algebra is a* $\boxdot$*-modal algebra.*

Theorem 1. *For every modal system* M*,* $\Phi \vdash_{\mathsf{M}} \varphi$ *iff* $\mathbf{L}^{\Phi}_{\mathsf{M}} \vDash \varphi \approx \top$.

The algebraic completeness of a modal system M w.r.t. a corresponding subclass of $\boxdot$-modal algebras is obtained as a corollary of Thm. 1. In other words, an M-Lindenbaum-Tarski $\boxdot$-modal algebra acts as an 'algebraic canonical model' for a set of formulas in the modal system M, i.e., they provide a witness for $\Phi \nvdash_{\mathsf{M}} \varphi$. We make full use of M-Lindenbaum-Tarski $\boxdot$-modal algebras in Sec. 3.3.

3 Default Modal Logic

In this section we integrate the elements of Default Logic, defaults and extensions, into modal systems. This integration yields what we call a default modal system. For each default modal system, we introduce an associated notion of default consequence and show how defaults interact with the rules of the Hilbert-style notion of deduction for the underlying modal system. Moreover, we present how a default modal system can be viewed from an algebraic perspective, and prove a completeness result using algebraic tools. Later on, we discuss how the algebraic treatment of default modal systems can be seen as an update operation on algebraic structures. This opens up the door to thinking about default systems from a dynamic logic perspective (akin to public announcements).

3.1 Default Modal Systems

The main elements of Default Logic, i.e., defaults and extensions, are given in Defs. 12 and 13, respectively. These definitions are adapted from [20]. For the rest of this section we assume that M is an arbitrary but fixed modal system.

Definition 12. *A default is a triple* (π, ρ, χ) *of formulas written as* $\pi : \rho \,/\, \chi$. *The formulas* π, ρ, *and* χ, *are called prerequisite, justification, and consequent.*

Definition 13. *Let* Φ *be a set of formulas and* Δ *a set of defaults. Let* $\mathsf{E}^{\Phi}_{\Delta\mathsf{M}}$ *be a function s.t. for all sets of formulas* Ψ, $\mathsf{E}^{\Phi}_{\Delta\mathsf{M}}(\Psi)$ *is the* $\subseteq$*-smallest set of formulas which satisfies:*

(a) $\Phi \subseteq \mathsf{E}^{\Phi}_{\Delta\mathsf{M}}(\Psi)$;

(b) $\mathsf{E}^{\Phi}_{\Delta\mathsf{M}}(\Psi) = \{\, \psi \mid \mathsf{E}^{\Phi}_{\Delta\mathsf{M}}(\Psi) \vdash_{\mathsf{M}} \psi \,\}$;

(c) for all $\pi : \rho \,/\, \chi \in \Delta$, *if* $\pi \in \mathsf{E}^{\Phi}_{\Delta\mathsf{M}}(\Psi)$ *and* $\neg\rho \notin \Psi$, *then,* $\chi \in \mathsf{E}^{\Phi}_{\Delta\mathsf{M}}(\Psi)$.

A set $\mathrm{E} \subseteq \mathsf{Form}$ *is an* M-extension *of* Φ *under* Δ *iff it is a fixed point of* $\mathsf{E}^{\Phi}_{\Delta}$, *i.e., iff* $\mathrm{E} = \mathsf{E}^{\Phi}_{\Delta}(\mathrm{E})$. *We write* $\mathscr{E}^{\Phi}_{\Delta\mathsf{M}}$ *for the set of all* M*-extensions of* Φ *under* Δ.

Intuitively, an M-extension can be thought of as a set of formulas which contains Φ, is closed under $\vdash_{\mathsf{M}}$, and is saturated under the application of the defaults in Δ. When it can be clearly understood from the context, we will drop the prefix M and refer to an M-extension as an extension.

In the literature on Default Logic, defaults are intuitively understood as defeasible rules of inference, i.e., rules of inference whose conclusions are subject to annulment, or rules which allow us to "jump" to conclusions. In turn, extensions are intuitively understood as sets of formulas closed under the application of defaults. The next two examples illustrate two properties of extensions: multiplicity and absence of extensions.

Example 1. Let $\Phi = \{\Diamond p\}$ and $\Delta = \{\Diamond p : \Diamond\neg p \,/\, \Diamond\neg p, \Diamond p : \Box p \,/\, \Box p\}$; the set $\mathscr{E}^{\Phi}_{\Delta\mathsf{M}}$ of extensions of Φ under Δ consists of exactly two extensions: (1) the set $\mathrm{E}_1 = \{\, \varphi \mid \{\Diamond p, \Diamond\neg p\} \vdash_{\mathsf{M}} \varphi \,\}$; and (2) the set $\mathrm{E}_2 = \{\, \varphi \mid \{\Diamond p, \Box p\} \vdash_{\mathsf{M}} \varphi \,\}$.

Each of the extensions in Ex. 1 corresponds to the application of each default in Δ. Once one default has been applied, the application of the other one is blocked. This example illustrates how to handle contradictory information.

Example 2. Let $\Phi = \{\Diamond p\}$ and $\Delta = \{\Diamond p : \Diamond q \,/\, \Box\neg q\}$; the set $\mathscr{E}^{\Phi}_{\Delta\mathsf{M}}$ of extensions of Φ under Δ is empty, i.e., $\mathscr{E}^{\Phi}_{\Delta\mathsf{M}} = \emptyset$, i.e., there are no extensions of Φ under Δ.

Ex. 2 highlights a subtletly in thinking of extensions as being constructed by the successive application of defaults: applying a default may result in its own annulment. To make this point clear, w.l.o.g., notice that plausible candidates for extensions are: the set $\mathrm{E}_1 = \{ \varphi \mid \{\Diamond p\} \vdash_{\mathsf{M}} \varphi \}$ (i.e., not applying the default); or the set $\mathrm{E}_2 = \{ \varphi \mid \{\Diamond p, \Box\neg q\} \vdash_{\mathsf{M}} \varphi \}$ (i.e., result of applying the default to E_1). Neither of these sets is a fixed point of $\mathsf{E}^{\Phi}_{\Delta}$, i.e., $\mathsf{E}^{\Phi}_{\Delta}(\mathrm{E}_1) = \mathrm{E}_2$ and $\mathsf{E}^{\Phi}_{\Delta}(\mathrm{E}_2) = \mathrm{E}_1$. This results in $\mathscr{E}^{\Phi}_{\Delta\mathsf{M}} = \emptyset$.

We are now in a position to define what we mean by a default modal system. This definition arises as a natural construction over a modal system M.

Definition 14. *A default modal system is a tuple $\Delta\mathsf{M} = \langle \Delta, \mathsf{M} \rangle$ where Δ is a set of defaults and M is a modal system.*

In analogy with the case in modal systems, we associate with each default modal system $\Delta\mathsf{M}$ a relation $\Vdash\!\sim_{\Delta\mathsf{M}}$ between sets of formulas and formulas. This relation is based on the relation $\vdash_{\mathsf{M}}$ and it can be understood as its default version. This is made clear in Def. 15.

Definition 15. *Let $\Delta\mathsf{M}$ be a default modal system; define*

$$\Phi \Vdash\!\sim_{\Delta\mathsf{M}} \varphi \quad \textit{iff} \quad \varphi \in \mathrm{E} \textit{ for some } \mathrm{E} \in \mathscr{E}^{\Phi}_{\Delta\mathsf{M}}.$$

We use $\Vdash\!\sim_{\Delta\mathsf{M}} \varphi$ as a shorthand for $\emptyset \Vdash\!\sim_{\Delta\mathsf{M}} \varphi$. The relation $\Vdash\!\sim_{\Delta\mathsf{M}}$ is called *credulous* in the literature on Default Logic, because the existence of just one extension is enough to grant the inference (see [2]). The principle of *monotonicity* fails for $\Vdash\!\sim_{\Delta\mathsf{M}}$. In other words: it is not necessarily the case that if $\Phi \Vdash\!\sim_{\Delta\mathsf{M}} \varphi$, then $\Phi \cup \Psi \Vdash\!\sim_{\Delta\mathsf{M}} \varphi$ (for an arbitrary Ψ).

Building the relation $\Vdash\!\sim_{\Delta\mathsf{M}}$ on the underlying relation $\vdash_{\mathsf{M}}$ raises the question of which properties of $\vdash_{\mathsf{M}}$ are preserved at the level of $\Vdash\!\sim_{\Delta\mathsf{M}}$. Def. 16 sets a basis on which to start answering this question.

Definition 16. *The relation $\Vdash\!\sim_{\Delta\mathsf{M}}$ interprets $\vdash_{\mathsf{M}}$ iff if $\Phi \vdash_{\mathsf{M}} \varphi$ then $\Phi \Vdash\!\sim_{\Delta\mathsf{M}} \varphi$.*

Interpretability seems to be a natural requirement on $\Vdash\!\sim_{\Delta\mathsf{M}}$. However, as established in Ex. 2 (which shows that sometimes extensions do not exist) this property fails to hold in general. To overcome this problem we can go down two possible paths: (i) modify Def. 13 to guarantee the existence of extensions; or (ii) single out defaults for which extensions are guaranteed to exist. Among the most popular modifications of Def. 13 which guarantee the existence of extensions we have: *justified* extensions (see [16]); and *constrained* extensions (see [5]). For option (ii), we have the set of *well-behaved*[5] defaults as a very large and natural set which guarantees the existence of extensions (see [20]). Going down path (i) overburdens the definition of an extension with additional machinery which departs from the purposes of our work here. For this reason, we choose to go down path (ii); i.e., we restrict ourselves to well-behaved defaults. Interestingly, the notions of extensions, justified extensions, and constrained extensions, coincide for well-behaved defaults (see [10,4]).

[5] In the literature on Default Logic well-behaved defaults are called normal. We avoid using this terminology here to avoid any confusion with normality in Modal Logic.

Definition 17. *A default $\pi : \rho \,/\, \chi$ is well-behaved, written π/χ, iff $\rho = \chi$. A set of defaults Δ is well-behaved iff all defaults in Δ are well-behaved. A default modal system $\Delta\mathsf{M}$ is well-behaved iff Δ is well-behaved.*

Proposition 5. *Let $\Delta\mathsf{M}$ be a default modal system; if $\Delta\mathsf{M}$ is well-behaved, then, $\Vdash\!\sim_{\Delta\mathsf{M}}$ interprets $\vdash_{\mathsf{M}}$.*

We conclude this section by drawing attention to an interesting point regarding necessitation in default modal systems in Prop. 6 (cf. item 1 in Prop. 1).

Proposition 6. *If $\Vdash\!\sim_{\Delta\mathsf{M}} \varphi$, then $\Vdash\!\sim_{\Delta\mathsf{M}} \Box\varphi$.*

Proof. Suppose that $\Vdash\!\sim_{\Delta\mathsf{M}} \varphi$; by definition, there is an M-extension $\mathrm{E} \in \mathscr{E}^{\Phi}_{\Delta\mathsf{M}}$ s.t. $\mathrm{E} \vdash_{\mathsf{M}} \varphi$. It follows that $\mathrm{E} \vdash_{\mathsf{M}} \Box\varphi$. Thus, $\Vdash\!\sim_{\Delta\mathsf{M}} \Box\varphi$.

The analogous to item 2 in Prop. 1, a form of the deduction theorem, i.e., if $\Phi \cup \{\varphi\} \Vdash\!\sim_{\Delta\mathsf{M}} \psi$, then, $\Phi \Vdash\!\sim_{\Delta\mathsf{M}} \boxdot\varphi \to \psi$ fails to hold for an arbitrary $\Delta\mathsf{M}$ (even with the presense of $\boxdot$).

3.2 Deducibility in Default Modal Systems

We formulate a notion of $\Delta\mathsf{M}$-deduction for an arbitrary but fixed well-behaved default modal system $\Delta\mathsf{M}$. This notion of a $\Delta\mathsf{M}$-deduction extends that of an M-deduction by incorporating defaults in a natural way.

Definition 18. *A $\Delta\mathsf{M}$-deduction of φ from Φ is a finite sequence $\psi_1 \dots \psi_n$ of formulas of* Form *s.t. $\psi_n = \varphi$, and for each $k < n$:*

1. *ψ_k is an axiom of* M*;*
2. *ψ_k is a premiss, i.e., $\psi_k \in \Phi$;*
3. *ψ_k is obtained from two earlier formulas using* mp*, i.e., there are $i, j < k$ s.t. $\psi_j = \psi_i \to \psi_k$;*
4. *ψ_k is obtained from an earlier formula using* u*, i.e., there is $j < k$ s.t. $\psi_k = \boxdot\psi_j$.*
5. *ψ_k is obtained from an earlier formula using Δ-detachment, i.e., there is $j < k$ s.t. $\psi_j/\psi_k \in \Delta$;*

A $\Delta\mathsf{M}$-deduction is credulous *whenever:*

$$(\Phi \cup \{\, \psi_i \mid 1 \le i \le n \,\}) \vdash_{\mathsf{M}} \bot \quad \textit{iff} \quad \Phi \vdash_{\mathsf{M}} \bot. \tag{1}$$

We define $\Phi \vdash\!\sim_{\Delta\mathsf{M}} \varphi$ iff there is a credulous $\Delta\mathsf{M}$-deduction of φ from Φ.

The notion of a credulous $\Delta\mathsf{M}$-deduction extends the notion of M-deduction with a rule of default detachment and the condition of being credulous. The rule of default detachment shows how defaults interact with the rules of the underlying proof system. Intuitively, a credulous $\Delta\mathsf{M}$-deduction of φ from Φ internalizes the construction of (part of) an extension containing φ together with the M-deduction which witnesses this containment. This is made precise in the following result.

Theorem 2. *$\Phi \vdash\!\sim_{\Delta\mathsf{M}} \varphi$ iff $\Phi \Vdash\!\sim_{\Delta\mathsf{M}} \varphi$.*

3.3 Towards an Algebraic Treatment of Default Modal Systems

We turn now our attention to a characterization of defaults and extensions by means of Lindenbaum-Tarski $\boxdot$-modal algebras. This algebraic treatment of defaults and extensions reveals how default modal systems may be thought of as updates on $\boxdot$-modal algebras. For the rest of this section, we assume that $\Delta\mathsf{M}$ is an arbitrary but fixed well-behaved default modal system. We use $\mathfrak{L}$ to indicate the class of Lindenbaum-Tarski $\boxdot$-modal algebras of M, i.e., $\mathfrak{L} = \{ \mathbf{L}^{\Phi}_{\mathsf{M}} \mid \Phi \subseteq \mathsf{Form} \}$. We drop the sub-index M and use Φ instead of $\cong^{\Phi}_{\mathsf{M}}$ as a way of further simplifying the notation. We construct this section around the following definition.

Definition 19. *Let $\delta = \pi/\chi \in \Delta$; the function $\hat{\delta} : \mathfrak{L} \to \mathfrak{L}$ is defined as:*

$$\hat{\delta}(\mathbf{L}^{\Phi}) = \begin{cases} \mathbf{L}^{\Phi\cup\{\chi\}} & \text{if } [\pi]_{\Phi} = 1_{\Phi} \text{ and } 0_{\Phi} \notin \uparrow\{[\boxdot\chi]_{\Phi}\} & \text{(2a)} \\ \mathbf{L}^{\Phi} & \text{otherwise.} & \text{(2b)} \end{cases}$$

Def. 19 is the algebraic counterpart of the application of a default w.r.t. a set of sentences. More precisely, $\delta = \pi/\chi$ is applicable w.r.t. a set Φ satisfying $\Phi = \{ \varphi \mid \Phi \vdash \varphi \}$ if: (a) $\pi \in \Phi$; and (b) $\Phi \cup \{\chi\} \nvdash \bot$. Applying the default δ results in $\{ \varphi \mid \Phi \cup \{\chi\} \vdash \varphi \}$. On the algebraic side, we capture the application of a default as a transformation between Lindenbaum-Tarski $\boxdot$-modal algebras. More precisely, consider the Lindenbaum-Tarski $\boxdot$-modal algebra for Φ, i.e., $\mathbf{L}^{\Phi}$. The condition (a) of applicability of $\delta = \pi/\chi$ w.r.t. $\mathbf{L}^{\Phi}$ is captured in (2a) as $[\pi]_{\Phi} = 1_{\Phi}$; and the condition (b) of applicability is captured in (2a) as $0_{\Phi} \notin \uparrow\{[\boxdot\chi]_{\Phi}\}$. In other words, the equivalence class of 1_{Φ} captures the deducibility of π from Φ, i.e., $\pi \in \Phi$, alt., $\Phi \vdash \pi$. In turn, the condition of *being proper* on the (open) filter generated by $[\boxdot\chi]_{\Phi}$ captures the consistency of χ w.r.t. Φ, i.e., $\Phi \cup \{\chi\} \nvdash \bot$. Notice that if the default is applicable, the return value of $\hat{\delta}$ incorporates χ to $\mathbf{L}^{\Phi}$, i.e., it results in $\mathbf{L}^{\Phi\cup\{\chi\}}$. Otherwise, $\hat{\delta}$ has no effect on $\mathbf{L}^{\Phi}$. When seen in this light, the operator $\hat{\delta}$ performs an *update* reflecting the application of δ on its input. The situation with $\hat{\delta}$ is similar to the case in dynamic logics such as Public Announcement Logic [19] (in particular, in relation to the approach proposed in [17]). We retake this discussion in Sec. 4.

Having dealt with defaults we turn our attention to extensions. For well-behaved defaults, extensions can be seen as being constructed in a step-wise fashion applying defaults one at a time. From a syntactic perspective, this construction of an extension starts with a closed set Φ, and applies the defaults $\delta \in \Delta$ one by one until we obtain a closed set of formulas that is saturated under the application of defaults. From the perspective of Lindenbaum-Tarski $\boxdot$-modal algebras we obtain the following.

Proposition 7. *Each function $\hat{\delta}$ induces a function $\bar{\delta} : |\mathbf{L}| \to |\hat{\delta}(\mathbf{L})|$ defined as: $\bar{\delta}([\varphi]_{\Phi}) = [\varphi]_{\Phi\cup\{\chi\}}$ if Eq.* (2a) *holds; or $\bar{\delta}([\varphi]_{\Phi}) = [\varphi]_{\Phi}$ if Eq.* (2b) *holds. The function $\bar{\delta}$ is a homomorphism from $\mathbf{L}$ to $\hat{\delta}(\mathbf{L})$.*

Proof. That $\bar{\delta}$ is a function is trivial. The proof that $\bar{\delta}$ is a homomorphism is by cases. If Eq. (2b) holds, then, the result is obtained immediately. Otherwise:

$$\bar{\delta}(f^{\Box}_{\Phi}([\varphi]_{\Phi})) = \bar{\delta}([\Box\varphi]_{\Phi}) = [\Box\varphi]_{\Phi\cup\{\chi\}} = f^{\Box}_{\Phi\cup\{\chi\}}([\varphi]_{\Phi\cup\{\chi\}}) = f^{\Box}_{\Phi\cup\{\chi\}}(\bar{\delta}([\varphi]_{\Phi})).$$

The remaining cases are similar.

The following are some immediate properties of default operators.

Definition 20. *Let* $\mathbf{L}_1, \mathbf{L}_2 \in \mathfrak{L}$*; we write* $\mathbf{L}_1 \leq \mathbf{L}_2$ *iff there is a homomorphism* $h : \mathbf{L}_1 \to \mathbf{L}_2$*; and* $\mathbf{L}_1 < \mathbf{L}_2$ *iff* $\mathbf{L}_1 \leq \mathbf{L}_2$ *and* $\mathbf{L}_1$, $\mathbf{L}_2$ *are not isomorphic.*

Proposition 8. *Every* $\hat{\delta}$ *is extensive and idempotent, i.e., it satisfies* $\mathbf{L} \leq \hat{\delta}(\mathbf{L})$ *and* $\hat{\delta}(\mathbf{L}) = \hat{\delta}(\hat{\delta}(\mathbf{L}))$*, resp. An arbitrary* $\hat{\delta}$ *needs not satisfy monotonicity, i.e., there are* $\delta = \pi/\chi$ *s.t.* $\mathbf{L}_1 \leq \mathbf{L}_2$ *and* $\hat{\delta}(\mathbf{L}_1) \not\leq \hat{\delta}(\mathbf{L}_2)$*.*

Proof. Extensivity follows from Prop. 7. Idempotence is proven by cases. If Eq. (2b) holds, then, the result is obtained immediately. Otherwise, Eq. (2a) holds. In this case, $\hat{\delta}(\mathbf{L}^{\Phi}) = \mathbf{L}^{\Phi\cup\{\chi\}}$. Trivially, $\hat{\delta}(\mathbf{L}^{\Phi\cup\{\chi\}}) = \mathbf{L}^{\Phi\cup\{\chi\}}$. For a counter-example to monotonicity consider $\mathbf{L}^{\emptyset}_{\mathsf{K}^{\boxdot}}$ and $\mathbf{L}^{\{\Box p\}}_{\mathsf{K}^{\boxdot}}$, and $\delta = \top/\Diamond\neg p$.

The set Δ of defaults leads naturally to a set $\{\, \hat{\delta} : \mathfrak{L} \to \mathfrak{L} \mid \delta \in \Delta \,\}$. Each $\hat{\delta}$ in this set can be seen as "taking a step" in the construction of the algebraic counterpart of an extension. To carry out this construction, we would need to compose such steps. This leads to the formulation of Def. 21.

Definition 21. *The default monoid associated to* $\Delta\mathsf{M}$ *is the monoid* $\mathbf{D}^*$ *freely generated by* $\{\, \hat{\delta} \mid \delta \in \Delta \,\}$*, i.e.,* $\mathbf{D}^* = \langle \mathrm{D}, -;-, \mathrm{id} \rangle$ *where:*

1. D *is the* $\subseteq$*-smallest set s.t.:* $\{\, \hat{\delta} : \mathfrak{L} \to \mathfrak{L} \mid \delta \in \Delta \,\} \subseteq \mathrm{D}$*;* $\mathrm{id} : \mathfrak{L} \to \mathfrak{L} \in \mathrm{D}_\Delta$*; if* $\{d_1 : \mathfrak{L} \to \mathfrak{L}, d_2 : \mathfrak{L} \to \mathfrak{L}\} \subseteq \mathrm{D}_\Delta$*, then* $(d_1;d_2) : \mathfrak{L} \to \mathfrak{L} \in \mathrm{D}_\Delta$*; and*
2. id *and* $-;-$ *satisfy:* $\mathrm{id}(\mathbf{L}) = \mathbf{L}$*; and* $(d_1;d_2)(\mathbf{L}) = d_2(d_1(\mathbf{L}))$*.*

Proposition 9. *Every* $d \in |\mathbf{D}^*|$ *is either: the identity, i.e.,* $d = \mathrm{id}$*; or a composition of the form* $d = (\hat{\delta}_1; \ldots ;\hat{\delta}_n)$*, where* $\delta_i \in \Delta$*.*

We define $\overline{\mathrm{id}}([\varphi]_\Phi) = [\varphi]_\Phi$; and $\overline{(\hat{\delta}_1; \ldots ;\hat{\delta}_n)} = (\bar{\delta}_1; \ldots ;\bar{\delta}_n)$.

Definition 22. *Let* $\mathbf{L}$ *be a Lindenbaum-Tarski* $\boxdot$*-modal algebra in* $\mathfrak{L}$*, and* v *be an assignment on* $\mathbf{L}$*; for an equation* $\varphi \approx \psi$*, define:*

$$\mathbf{D}^*, \mathbf{L}, v \approx\!\!\!\models \varphi \approx \psi \quad \text{iff} \quad d(\mathbf{L}), (v;\bar{d}) \models \varphi \approx \psi \text{ for some } d \in |\mathbf{D}^*|.$$

We write $\mathbf{D}^*, \mathbf{L} \approx\!\!\!\models \varphi \approx \psi$ *iff* $\mathbf{D}^*, \mathbf{L}, v \approx\!\!\!\models \varphi \approx \psi$ *for all assignments* v*.*

Intuitively, the Lindenbaum-Tarski $\boxdot$-modal algebra $d(\mathbf{L})$ in Def. 22 is the algebraic counterpart of the concept of an extension. This is made clear in Thm. 3.

Theorem 3. $\Phi \mathrel{|\!\sim} \varphi$ *iff* $\mathbf{D}^*, \mathbf{L}^{\Phi} \approx\!\!\!\models \varphi \approx \top$*.*

Proof. The interesting part is the right-to-left implication: if $\mathbf{D}^*, \mathbf{L}^{\Phi} \approx\!\!\!\models \varphi \approx \top$, then, $\Phi \mathrel{|\!\sim} \varphi$. We prove the contrapositive: if $\Phi \mathrel{|\!\not\sim} \varphi$, then, $\mathbf{D}^*, \mathbf{L}^{\Phi} \not\approx\!\!\!\models \varphi \approx \top$. Let $\Phi \mathrel{|\!\not\sim} \varphi$, the proof is concluded if for all $d \in |\mathbf{D}^*|$, $d(\mathbf{L}^{\Phi}) \not\models \varphi \approx \top$. We continue by induction on d. Let $d = \mathrm{id}$; we must have $\mathrm{id}(\mathbf{L}^{\Phi}) \not\models \varphi \approx \top$; otherwise we would obtain $\Phi \vdash \varphi$ (from Thm. 1); and so that $\Phi \mathrel{|\!\sim} \varphi$ (which contradicts our assumption). For the next case, let $d = \hat{\delta}$ for $\delta = \pi/\chi \in \Delta$; either Eq. (2b) holds or Eq. (2a) holds. If Eq. (2b) holds, $\hat{\delta}$ behaves like id (and we are back to the previous case). If Eq. (2a) holds, $\hat{\delta}(\mathbf{L}^{\Phi}) = \mathbf{L}^{\Phi\cup\{\chi\}}$. Assuming (i) $\mathbf{L}^{\Phi\cup\{\chi\}} \models \varphi \approx \top$ leads to a contradiction. More precisely, if Eq. (2a) holds, from Thm. 1, we obtain $\Phi \vdash \pi$ and $\Phi \cup \{\chi\} \nvdash \bot$. From (i) and Thm. 1, we obtain $\Phi \cup \{\chi\} \vdash \varphi$. If we place the M-deduction of π from Φ in front of the M-deduction of φ from $\Phi\cup\{\chi\}$, we obtain $\Phi \vdash \varphi$. This yields the contradiction. For the inductive step, let $d = (\hat{\delta}_1; \ldots; \hat{\delta}_n; \hat{\delta}_{(n+1)})$. Suppose that $(\hat{\delta}_1; \ldots; \hat{\delta}_n)(\mathbf{L}^{\Phi}) = \mathbf{L}^{\Phi'}$. From the inductive hypothesis, we obtain $\mathbf{L}^{\Phi'} \not\models \varphi \approx \top$. Assuming that $\hat{\delta}_{(n+1)}(\mathbf{L}^{\Phi'}) \models \varphi \approx \psi$ leads to a contradiction using the same argument as in (i).

We conclude this section by taking some steps beyond dealing with defaults and extensions in the context of Lindenbaum-Tarski $\boxdot$-modal algebras. In particular, we show how some of the constructions used in Sec. 3.3 can be extended to a more abstract setting via suitable congruences.

Definition 23. *Let $\mathbf{L}^{\Phi}$ be a Lindenbaum-Tarski $\boxdot$-modal algebra and χ a formula; define $[\varphi_1]_{\Phi} \equiv_{\chi} [\varphi_2]_{\Phi}$ iff $[\varphi_1]_{\Phi} *_{\Phi} [\boxdot\chi]_{\Phi} = [\varphi_2]_{\Phi} *_{\Phi} [\boxdot\chi]_{\Phi}$.*

Def. 23 is a step towards treating the application of default as a device for obtaining a $\boxdot$-modal algebra $\mathbf{M}$ updated by the element $[\chi]_{\Phi}$ in $\mathbf{L}^{\Phi}$. The updated $\boxdot$-modal algebra $\mathbf{M}$ is meant to be obtained as a quotient algebra modulo the congruence $\equiv_{\chi}$. Prop. 10 shows that $\equiv_{\chi}$ indeed is a congruence.

Proposition 10. *The relation $\equiv_{\chi}$ is a congruence on $\mathbf{L}^{\Phi}$.*

Proof. That $\equiv_{\chi}$ is an equivalence relation is immediate. To improve notation we drop the subscript $_{\Phi}$. We need to show that: if $[\varphi_1] \equiv_{\chi} [\varphi_2]$ and $[\varphi_3] \equiv_{\chi} [\varphi_4]$, then, $[\varphi_1] + [\varphi_3] \equiv_{\chi} [\varphi_2] * [\varphi_4]$; $-[\varphi_1] \equiv_{\chi} -[\varphi_2]$; $f^{\Box}([\varphi_1]) \equiv_{\chi} f^{\Box}([\varphi_2])$; and $f^{\boxdot}([\varphi_1]) \equiv_{\chi} f^{\boxdot}([\varphi_2])$. The proof continues by cases.

$$\begin{aligned} & f^{\Box}([\varphi_1]) * [\boxdot\chi] \\ & \quad \geq f^{\Box}([\varphi_1] * [\boxdot\chi]) * [\boxdot\chi] \\ & \quad = f^{\Box}([\varphi_2] * [\boxdot\chi]) * [\boxdot\chi] \\ & \quad = f^{\Box}([\varphi_2]) * (f^{\Box}([\boxdot\chi]) * [\boxdot\chi]) \\ & \quad \geq f^{\Box}([\varphi_2]) * [\boxdot\chi] \end{aligned}$$

$$\begin{aligned} & f^{\boxdot}([\varphi_1]) * [\boxdot\chi] \\ & \quad = f^{\boxdot}([\varphi_1]) * [\boxdot\boxdot\chi] \\ & \quad = f^{\boxdot}([\varphi_1]) * f^{\boxdot}([\boxdot\chi]) \\ & \quad = f^{\boxdot}([\varphi_1] * [\boxdot\chi]) \\ & \quad = f^{\boxdot}([\varphi_2] * [\boxdot\chi]) \\ & \quad = f^{\boxdot}([\varphi_2]) * f^{\boxdot}([\boxdot\chi]) \\ & \quad = f^{\boxdot}([\varphi_2]) * [\boxdot\boxdot\chi] \\ & \quad = f^{\boxdot}([\varphi_2]) * [\boxdot\chi]. \end{aligned}$$

The remaining cases are routine.

Proposition 11. *The quotient algebra $\mathbf{L}^{\Phi}/_{\equiv_{\chi}}$ is isomorphic to $\mathbf{L}^{\Phi\cup\{\chi\}}$.*

Proof (sketch). Observe that $\Phi \cup \{\chi\} \vdash (\varphi_1 \leftrightarrow \varphi_2)$ iff $\Phi \vdash (\varphi_1 \wedge \boxdot\chi \leftrightarrow \varphi_2 \wedge \boxdot\chi)$. The isomorphism between $\mathbf{L}^{\Phi}/_{\equiv_{\chi}}$ and $\mathbf{L}^{\Phi\cup\{\chi\}}$ is given by mappings ι_1 and ι_2 defined as: $\iota_1([[\varphi]_{\Phi}]_{\equiv_{\chi}}) = [\varphi]_{\Phi\cup\{\chi\}}$; and $\iota_2([\varphi]_{\Phi\cup\{\chi\}}) = [[\varphi]_{\Phi}]_{\equiv_{\chi}}$.

The isomorphism in Prop. 11 shows that the relation $\equiv_\chi$ yields the "correct" congruence if the application of a default is to be seen as an update on a $\boxdot$-modal algebra. Moreover, it is possible to define a function $\varepsilon : \mathbf{L}^\Phi/_{\equiv_\chi} \to \mathbf{L}^\Phi$ defined by $\varepsilon([[\varphi]_\Phi]_{\equiv_\chi}) = [\varphi]_\Phi *_\Phi [\chi]_\Phi$. The image of ε is also isomorphic to $\mathbf{L}^{\Phi\cup\{\chi\}}$. The results discussed in this paragraph open a pathway on how to lift the constructions in Defs. 19 and 21 to the setting of arbitrary $\boxdot$-modal algebras.

4 On Defaults as Model Updates

We are now in a position to establish a connection between our algebraic approach for default modal systems and the algebraic treatment of Public Announcement Logic (PAL) in [17]. To set up context for discussion, we briefly introduce some basic notions of PAL (see, e.g., [19] for details). As a modal logic, PAL extends the modal logic S5 (seen as the logic of knowledge) with a new modality $\langle !\psi\rangle$ of announcement. Intuitively, $\langle !\psi\rangle\varphi$ states that after the truthful announcement of ψ, φ holds. Model theoretically, the interpretation of announcing ψ relativizes the model in which ψ is announced to the submodel in which ψ holds. The formula φ is then evaluated on the relativized model. Notice that the announcement of ψ must be truthful: it occurs only if ψ is true. Otherwise, the announcement fails and $\langle !\psi\rangle\varphi$ is evaluated as false.

There are some interesting similarities between announcements in PAL and defaults. From an algebraic perspective, an announcement may be understood as a homomorphism between the modal algebra in which the announcement occurs and the modal algebra corresponding to the submodel in which the announced formula holds. The algebraic machinery introduced in Sec. 3.3 sets the basis for thinking about the application of defaults as a logic of updates between particular modal algebras (Lindenbaum-Tarski $\boxdot$-modal algebras). In other words, we may construe the algebraic semantics of a default as an update from the Lindenbaum-Tarski $\boxdot$-modal algebra in which the default is considered, and the Lindenbaum-Tarski $\boxdot$-modal algebra updated with the consequent of the default (if the default is applicable). Notice also that, just as the update in the case of an announcement takes place only if the formula being announced holds, a default update takes place only if the prerequisite of the default is provable and its justification does not yield an inconsistency. Both, in the case of an announcements and in the algebraic application of a default, the update is captured by a homomorphism from the original modal algebra to an updated modal algebra. There is, however, a catch: if the announcement of ψ is not truthful the whole formula $\langle !\psi\rangle\varphi$ amounts to a falsity; if the prerequisite of a default is not provable, or its justification is inconsistent in the modal algebra, the application of the default has no effect.

The similarities between announcements in PAL and defaults are even more apparent w.r.t. the proposal in [17]. This proposal exploits the duality between models and algebras in order to algebraize PAL. In particular, in [17], a formula ψ is interpreted as an element b in an S5 modal algebra $\mathbf{M} = \langle B, *, -, f^\Box\rangle$. The result of announcing this formula is a modal algebra constructed as a quotient modulo a congruence $\equiv_b$ defined as $b_1 \equiv_b b_2$ iff $b_1 * b = b_2 * b$. This congruence bears a close resemblance to the one we presented in Sec. 3.3. The main difference between this congruence and ours rests on the fact that the former is presented in the setting of S5, whereas ours is presented in a setting where global modal consequence is taken as the basis on which to build default modal systems. This said, the approach in [17] is more abstract than ours; since it considers arbitrary modal algebras and not just Lindenbaum-Tarski modal algebras.

Evidently, these are only some first steps in understanding the relationship between defaults and updates: both in terms of a full algebraization of default modal systems, and in terms of establishing a tight connection with logics of updates. Regarding the full algebraization of default modal systems, we would like to interpret the application of a default not as an update over Lindenbaum-Tarski $\boxdot$-modal algebras, but over arbitrary modal algebras. In this regard, the main challenge is how to generalize the way in which we capture the application of one default, and how to capture a sequence of applications of defaults to build extensions. Regarding a tight connection with logics of updates, we would like to establish a reduction result between a default modal system and a logic of announcement (or establishing a difference in expressive power between one and the other). These are preliminary thoughts which constitute an interesting direction for further work. Finally, upon defining the semantics of defaults as model updates, we would also like to study defaults as dynamic epistemic operators. In particular, it would be interesting to know whether a default represents some kind of communication between agents in a multi-agent setting.

5 Final Remarks

We presented a family of default logics built over modal logics, and studied some of their properties.

First, we presented default logics syntactically as a default modal system. For each default modal system we formulated a notion of default deducibility to make explicit how defaults interact with the rules of the underlying proof calculus. Then, we offered an algebraic treatment of defaults and extensions. The algebraic treatment enabled us to obtain an algebraic completeness result. To our knowledge, this is the first work addressing default logic algebraically.

Moreover, we discussed a connection between default modal systems and modal logics with updates. In particular, our algebraic treatment of defaults is inspired by the ideas introduced in [17] for PAL. We believe that considering default modal systems as logics of updates is an interesting pathway to the study of the meta-logical properties of such systems from a semantic perspective.

References

1. T. Ågotnes and N. Alechina. The dynamics of syntactic knowledge. *Journal of Logic and Computation*, 17(1):83–116, 2007.

2. G. Antoniou and K. Wang. Default logic. In D. Gabbay and J. Woods, editors, *The Many Valued and Nonmonotonic Turn in Logic*, volume 8 of *Handbook of the History of Logic*, pages 517–555. North-Holland, 2007.
3. P. Blackburn, M. de Rijke, and Y. Venema, editors. *Modal Logic*. Cambridge University Press, 2002.
4. V. Cassano, R. Fervari, C. Areces, and P. Castro. Interpolation and beth definability in default logics. In *Proceedings of JELIA 2019*, volume 11468 of *LNCS*, pages 675–691. Springer, 2019.
5. J. Delgrande, T. Schaub, and W. Jackson. Alternative approaches to default logic. *Artificial Intelligence*, 70(1-2):167–237, 1994.
6. H. N. Duc. Reasoning about rational, but not logically omniscient, agents. *Journal of Logic and Computation*, 7(5):633–648, 1997.
7. R. Fervari and F. R. Velázquez-Quesada. Dynamic epistemic logics of introspection. In *Proceedings of DALI 2017*, volume 10669 of *LNCS*, pages 82–97. Springer, 2017.
8. R. Fervari and F. R. Velázquez-Quesada. Introspection as an action in relational models. *J. Log. Algebraic Methods Program.*, 108:1–23, 2019.
9. M. Fitting. Modal proof theory. In P. Blackburn, J. van Benthem, and F. Wolter, editors, *Handbook of Modal Logic*, pages 85–138. Elsevier, 2007.
10. C. Froidevaux and J. Mengin. Default logics: A unified view. *Computational Intelligence*, 10:331–369, 1994.
11. D. Gabbay, A. Kurucz, F. Wolter, and M. Zakharyaschev. *Many-Dimensional Modal Logics: Theory and Applications*. Elsevier, 2003.
12. S. Givant and P. Halmos. *Introduction to Boolean Algebras*. Undergraduate Texts in Mathematics. Springer, 2009.
13. V. Goranko and S. Passy. Using the universal modality: Gains and questions. *Journal of Logic and Computation*, 2(1):5–30, 1992.
14. D. Grossi and F. R. Velázquez-Quesada. Syntactic awareness in logical dynamics. *Synthese*, 192(12):4071–4105, 2015.
15. M. Jago. Epistemic logic for rule-based agents. *Journal of Logic, Language and Information*, 18(1):131–158, 2009.
16. W. Łukaszewicz. Considerations on default logic: An alternative approach. *Computational Intelligence*, 4:1–16, 1988.
17. M. Ma. Mathematics of public announcements. In *Proceedings of LORI 2011*, pages 193–205. Springer Berlin Heidelberg, 2011.
18. A. Mikitiuk and M. Truszczynski. Constrained and rational default logics. In *Proceedings of IJCAI'95*, pages 1509–1517, 1995.
19. J. Plaza. Logics of public communications. *Synthese*, 158(2):165–179, 2007.
20. R. Reiter. A logic for default reasoning. *AI*, 13(1-2):81–132, 1980.
21. T. Schaub. On constrained default theories. In *11th European Conference on Artificial Intelligence (ECAI'92)*, pages 304–308, 1992.
22. A. Solaki. Steps out of logical omniscience. Master's thesis, Institute for Logic, Language and Computation, Amsterdam, 2017. MoL-2017-12.
23. J. van Benthem and E. Pacuit. Dynamic logics of evidence-based beliefs. *Studia Logica*, 99(1):61–92, 2011.
24. J. van Benthem and F. R. Velázquez-Quesada. The dynamics of awareness. *Synthese (Knowledge, Rationality and Action)*, 177(Supplement 1):5–27, 2010.
25. M. Y. Vardi. On epistemic logic and logical omniscience. In *Proceedings of TARK 1986*, pages 293–305. Morgan Kaufmann.
26. F. R. Velázquez-Quesada. Explicit and implicit knowledge in neighbourhood models. In *Proceedings of LORI 2013 Workshop*, pages 239–252, 2013.
27. Y. Venema. Algebras and general frames. In Blackburn et al. [3], pages 263–333.