A Deontic Logic of Knowingly Complying

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ABSTRACT

We introduce a logic for representing the deontic notion of *knowingly complying* –associated to an agent's conciousness of taking a normative course of action for achieving a certain goal. Our logic features an operator for describing normative courses of actions, and another operator for describing what each agent knowingly complies with. We provide a sound and complete axiom system for our logic, and study the computational complexity of its satisfiability problem. Finally, we extend our logic with an additional operator for capturing the general abilities of the agents. This operator enables us to distinguish 'what agents can do' and 'what agents do according to norms'. For this extension, we also provide a sound and complete axiom system.

KEYWORDS

Deontic Logic, Knowingly Complying, Multiagent

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1 INTRODUCTION

Normative systems are ubiquitous in many disciplines –e.g., legal reasoning, computer science, knowledge representation, philosophy, etc. To be able to reason rigurously, and so logically, in and about normative systems is an imperative. One of the most prominent logical approaches to normative reasoning corresponds to Deontic Logics [4, 23, 30]. In brief, deontic logics are formalisms tailored to describing and reasoning about scenarios involving norms and related concepts [15, 16]. In this respect, they can be used, e.g., to determine if such normative scenarios are free of contradictions or, so-called, paradoxes.

Typically, deontic logics propose operators to speak about the obligations, permissions, and prohibitions, of some actors –generally called *agents*– involved in a certain scenario. It has been argued, e.g., in [30, 31], that these operators should range over the *actions* executed by the agents, rather than over propositions or states of affairs. Deontic logics with this characteristic are commonly known as *ought-to-do* deontic logics [1]. To a large extent, ought-to-do deontic logics focus their attention on the normative status of actions at some specific moment –i.e., in a particular static situation. This view ignores for instance, the circumstances that lead an agent to reach a certain state, and overlooks the *course of action* that an agent may take to achieve a goal. To deal with these situations, ought-to-do deontic logics incorporate ideas coming from the world of *strategic* reasoning; see, e.g., [13, 22]. In a strategic ought-to-do deontic logic, the description of a normative situation considers not only the status of a given set of norms, but also provides a view on the behaviour of the agents with respect to their background.

In ought-to-do deontic logics, strategic reasoning is often achieved by incorporating ways of describing sequences of actions (or plans) over a temporal dimension. This temporal dimension is then used to model and study (a part of) a particular normative system. Many existing frameworks of this kind extend STIT logic [5] with temporal operators. This approach can be found, e.g., in [10, 11]. Therein, temporal operators for 'historical necessity', together with a standard epistemic operator of 'knowing that', are incorporated into a STIT logic. The resulting logical system is shown to accommodate for the notion of *knowingly doing* –whose goal is to characterize different modes of responsibility for an agent who breaks a norm (a concept known as 'guilty mind' or 'mens rea'). The work in [12] takes a similar approach but incorporates also an operator of 'intention' to represent different levels of culpability. Finally, [25] investigates the so-called T-STIT logic. T-STIT logic extends STIT logic with future and past tense operators, and with a group agency operator for coalitions involving all agents. The obtained logic is used to model normative concepts such as achievement obligation and commitment (see [22, Ch. 7], for details).

The logical frameworks mentioned above are highly expressive. This should not come as surprise, as these logics combine the expressivity of several operators of very different nature. In spite of this fact, the formalization of a particular deontic concept uses just a small fragment of these logics featuring a high expressive power, while such an expressivity impacts negatively on the overall computational behaviour of the logic. For instance, the work reported in [3, 20] studies fragments of STIT logic whose satisfiability problem is undecidable. The work from [27, 28] identifies fragments with complexities in ExpTime and NExpTime. Interestingly, in the last work it is also proven that very restricted fragments of STIT logics (i.e., those obtained by removing the temporal dimension or by limiting nested negations) are decidable in NP.

In this article, we take a different approach. We chose a specific, important deontic concept involving strategic reasoning: the notion of *knowingly complying*, and define a specialized formalism to work with it. This enables us to obtain a very natural representation while maintaining an excellent computational behaviour. Arguably, the

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resulting logic is also very simple (and elegant) and leads itself well to a detailed study using already known techniques from modal logic [7, 9]. Our proposal takes inspiration from formalisms recently introduced in the epistemic study of the notion of *knowing how* (see, e.g., [2, 19, 32, 33]).

Contributions. We focus on representing the notion of *knowingly complying* in a deontic setting. We say that *"an agent knowingly complies with a certain goal φ, given a certain initial condition ψ"*, whenever the agent has a normative course of action that leads from every situation in which ψ holds, only to situations where φ holds. Moreover, the agent tells apart such courses of actions from others that are not between the limits of the law. In doing so, we detect three main ingredients to consider (the terminology used here is inspired by [6]). First, we have the agents' *abilities*, i.e., an account of what the agents are *able to do* (all the possible strategies or courses of action that an agent may take for achieving a goal). Second, we consider a set of *norms*, fixing what the agents *must* do. Norms are expressed as a set of legal courses of actions. Third, we take into account the *responsibilities* of the agents, given by each agent's *own judgement*. This enables us to determine each agent's awareness when complying with a norm or not. Remarkably, our framework does not rely on putting together separate existing features, and then coming up with a clever formalization. Instead, we will provide a semantics in which the above-mentioned concepts are internalized in the logic, and where each component interacts with the others to obtain the intended behaviour. This makes our framework different from existing approaches, instead of an alternative to them.

From a semantic perspective, we borrow and adapt ideas from the *epistemic knowing how logic* presented in [2]. Therein, an agent's 'know how' is interpreted over *labelled transitions sytems (*LTS*)*, extended with a notion of 'epistemic indistinguishability' at the level of plans (understood as finite sequences of basic actions). The relational part of the LTS provides the *ontic* information, i.e., the factual information that is common for all agents. The indistinguishability relation between plans provides instead the *epistemic* information for each agent, via their own perception about the real world. This approach is in line with standard epistemic logics [14, 21]. Here, we generalize this semantics, taking the LTS as the component representing the abilities available for all agents; a set of plans which establishes the set of normed courses of actions; and finally, an indistinguishability relation between plans that captures each agent's own perception about the actions she can take. Then, we introduce corresponding modalities that enable us to express deontic properties over this kind of models: the deontic modality $\mathsf{N}(\psi, \varphi)$ stating that *"in any situation in which ψ holds, it is possible to achieve the goal φ according to the norms"*; and the modality $\mathsf{Kc}_i(\psi, \varphi)$, for each agent i, that states that *"in any situation in which ψ holds, agent i knowingly complies with φ"*. We investigate the logical properties of this deontic logic, in particular we provide a sound and strongly complete axiom system and show that its satisfiability problem is NP-complete. Then, we extend the language with a modality S that enables us to express properties about the general abilities for the agents, and its interactions with the previously introduced modalities. Capturing abilities in the language enables us to distinguish between what agents can do vs. what agents do according to norms. For this extension we also provide an axiomatization.

Emergency Procedure

– Fire – Smoke – Explosion	keep calm Pull Fire Alarm, from a safe location Call 999 (Fire Brigade)

When Alarm Rings SHUT OFF GAS and POWER

Evacuate: close doors behind, use only stairs or ramps.
If unsafe to evacuate: shut door, block cracks, stay low near window.
If room door hot: keep door closed, stay put, stay low near window.

Figure 1: Fire Emergency Evacuation Plan

Outline. In Sec. 2 we introduce an example that illustrates the choice of the logical components of our work. In Sec. 3 we define the syntax and semantics of the logic of *knowingly complying*. We accompany our definitions with the formal counterpart of the example from Sec. 2. In Sec. 4 we introduce a sound and complete axiom system, whereas in Sec. 5 we provide a characterization of the complexity of deciding satisfiability. In Sec. 6 we extend the logic in Sec. 3 with a new modality for capturing abilities in the language, and provide and axiomatization for this logic. Lastly, in Sec. 7 we provide some final remarks.

2 A MOTIVATING EXAMPLE

We motivate the logical components of our work using a fire emergency evacuation plan (FEEP) as a running example. A FEEP is a written document detailing the actions to be taken in the event of a fire, and the arrangements for calling the fire brigade. Fig. 1 illustrates a typical FEEP.[1] In spite of its simplicity, this FEEP contains some important norms to be complied with. For example, in the event of a fire/smoke/explosion, it is the duty of every person to take the following course of action: sound the nearest fire alarm, move to a safe location, and call the Fire Brigade. Moreover, every person should arrange for evacuating the premises in the light of some basic risk assessment and other emergency precautions, e.g., closing doors behind, or staying put and blocking doors if they are hot. The FEEP assumes awareness of the layout of the premises, capacity of identifying key escape routes via exit signs, and the possibility of vacating the premises using only stairs and/or ramps (elevators are excluded due to potential electrical failure and/or power outage). Finally, the procedure establishes that every person must remain calm in any possible situation, even if they have taken a wrong course of action (e.g., take an elevator in case of a fire).

In order to formalize how certain agents would act in a case of a fire, we detect the following basic components:

abilities: account for what the agents are 'able to do' to achieve a goal. In the FEEP, for instance, agents are able to take the stairs to bring themselves from an unsafe to a safe place. Certain combinations should also be possible, e.g., the FEEP establishes that in a case of fire, the agents need to "pull the alarm, take stairs/ramps to evacuate, and finally call 999". These sequences, commonly called *plans*, is what we understand as a course of action. In our setting, these abilities will be characterized via *Labelled Transition Systems (*LTS*s)*.

[1] In fact, Fig. 1 takes inspiration from the FEEP obtainable from the Ocupational Safety section on McMaster University's website.

norms: account for 'legal' courses of actions. They reflect the fact that some courses of action in some situations are regulated by norms. In our FEEP example, the plan "pull the alarm, take the stairs (or the ramp) and call 999" is prescribed in the case of a fire, whereas "use the elevator" is preempted.

responsibilities: captures each agent's *own judgement* for complying with a norm. From the perspective of a particular agent, there might be certain courses of action that are indistinguishable from others (equivalent among themselves with respect to the current scenario). E.g., agent i may consider that exiting the building using stairs or ramps is fine, while evacuating the premises using elevators should be avoided (as indicated by the FEEP). While agent j, that ignores the FEEP, may take that exiting the premises, no matter how, is what matters. In our formal setting, we will want to express that agent i knowingly complies with the norms, whereas j does not. Moreover, some agents might even be unaware of certain courses of action being possible; e.g., the location of a particular exit point. To do so, we will define a suitable notion of "being indistinguishable" between plans.

Notice that each component, uses and refines the previous ones. This will become clearer in the next section. In the rest of the paper we develop the formal machinery behind the provided intuitions.

3 DEONTIC LOGIC OF KNOWINGLY COMPLYING

In this section, we introduce the language and the semantics of our Deontic Logic of Knowingly Complying (DLKc). We assume denumerable sets Prop for proposition symbols, and Act for basic action names, and that Agt is a non-empty finite set of agent names. Moreover, we assume all these sets are pairwise disjoint.

Definition 3.1. *The* language *of* DLKc*, i.e., its set of formulas, is determined by the following grammar:*

$$\varphi, \psi ::= p \mid \neg\varphi \mid \varphi \vee \psi \mid \mathsf{N}(\psi, \varphi) \mid \mathsf{Kc}_i(\psi, \varphi),$$

where: $p \in \mathsf{Prop}$ *and* $i \in \mathsf{Agt}$. *We use* $\bot, \top, \varphi \wedge \psi, \varphi \to \psi$*, and* $\varphi \leftrightarrow \psi$ *as abbreviations defined as usual. Intuitively, a formula* $\mathsf{N}(\psi, \varphi)$ *is read as:* "there is a normative course of action that brings about φ given ψ"*; and a formula* $\mathsf{Kc}_i(\psi, \varphi)$ *is read as:* "agent i knowingly complies with φ given ψ". *We also write* $\mathsf{A}\varphi$ *and read it as:* "φ holds anywhere"*; and* $\mathsf{E}\varphi$ *read it as:* "φ holds somewhere". *The connectives* A *and* E *are the* universal *and* existential *modalities [17]. As shown in Prop. 3.1, they are definable in terms of other connectives.*

Example 3.1. *Tab. 1 illustrates how to use formulas of* DLKc *to express some properties in the context of the example in Sec. 2.*

We introduce the structures on which to interpret the formulas of DLKc in a step by step fashion.

Definition 3.2 (Plans). *We denote* Act^* *the set of all (possibly-empty) finite sequences over* Act. *The elements of* Act^* *are called* plans. *We use* ϵ *to denote the empty sequence in* Act^* *(the* empty plan*). For* $\pi \in \mathsf{Act}^*$, $|\pi|$ *is the length of* π. *For* $0 \le k \le |\pi|$, π_k *is the initial segment of* π *of length* k *and* $\pi[k]$ *is the* k^{th} *element of* π.

Intuitively, the elements in Act can be understood in correspondence to basic actions, and those in Act^* in correspondence to courses of action.

Table 1: Formulas and their Intuitive Readings.

Formula	*Intuitive reading*
$\mathsf{A}(s \to \neg f)$	*In any safe location (s), there is no fire (f).*
$\mathsf{E}f$	*There is the possibility of a fire.*
$\mathsf{N}(f \wedge c, s)$	*There exists a normative course of action that brings any agent to a safe location in case of a fire, whenever there is the capacity (c) to do so.*
$\mathsf{Kc}_i(f \wedge c, s)$	*Agent i knowingly complies with the norms regulating reaching a safe location in the event of a fire (a.k.a., knows how to conform to the FEEP), provided also that there is a capacity (c) to do so.*

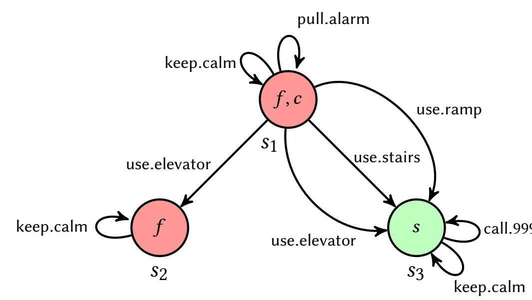

Figure 2: An LTS for the FEEP.

Example 3.2. *For the FEEP in Sec. 2, we may have basic action names such as:*

$$\begin{array}{lll} \mathsf{keep.calm}, & \mathsf{pull.alarm}, & \mathsf{call.999}, \\ \mathsf{use.stairs}, & \mathsf{use.ramp}, & \mathsf{use.elevator}, \quad \dots \end{array}$$

These basic actions names represent the actions of: remaining calm, pulling the fire alarm; calling the Fire Brigade; and using the stairs, a ramp, and an elevator to exit the building, resp. In turn, the plan

$$\pi_1 = \mathsf{pull.alarm}; \mathsf{use.stairs}; \mathsf{call.999}$$

indicates the course of action of pulling the fire alarm, evacuating the building using the stairs, and calling the Fire Brigade; while

$$\pi_2 = \mathsf{use.elevator}$$

would indicate the plan of using an elevator to exit the building.

In general, we will not be interested in all plans, but only on those that are somewhat delimited (i.e., we would like to rule out arbitrary arrangements of basic actions). Moreover, we may wish for certain plans to take place only in particular contexts, and with the purpose of bringing about some goal. These ideas are made precise in Def. 3.3.

Definition 3.3 (LTS). *A* labelled transition system (LTS) *is a tuple* $\mathfrak{L} = \langle \mathrm{S}, \mathrm{R}, \mathrm{V} \rangle$ *where:* S *is a non-empty set of states; for some* $A \subseteq \mathsf{Act}$, R *is an* A*-indexed family of binary relations on* S*, i.e.,* $\mathrm{R} = \{\mathrm{R}_a \subseteq \mathrm{S}^2 \mid a \in A\}$*; and* $\mathrm{V} : \mathrm{S} \to 2^{\mathsf{Prop}}$ *is an assignment function.*

Simply put, $\mathfrak{L}$ is a graph whose nodes are labelled with proposition symbols and whose edges are labelled with action names. In this way, V indicates the proposition symbols that hold on each state; whereas R represents –following the terminology in [32]– an *ability map*, i.e., the possible courses of action for all agents.

Table 2: Intuitive Interpretation of the states in Fig. 2.

state	*represents a situation in which:*
s_1	*A fire occurs, and there is the capacity to follow the evacuation protocol.*
s_2	*A fire occurs, but there is no capacity to follow the evacuation protocol (e.g., trapped in elevator).*
s_3	*A safe location has been reached, thus there is no fire.*

Example 3.3. *In Fig. 2 we present an* LTS *modelling a part of the FEEP in Sec. 2. The intuive interpretation of the states in this* LTS *is summarized in Tab. 2.*

Notice how the LTS adds contexts/goals (pre/post conditions) to basic actions, and thus to plans. We summarize what these contexts/goals look like for the plans in Ex. 3.2 in the table below.

plan	pre	post	plan	pre	post
π_1	$f \wedge c$	s	π_2	$f \wedge c$	$(f \wedge \neg c) \vee (s \wedge \neg f)$

Intuitively, the plan π_1 has as a context the occurrence of a fire and the capacity to follow the evacuation protocol, i.e., it is a fact that such a capacity exists; its goal is that of taking the agent to a safe place (and thus one in which there is no fire). In turn, plan π_2 takes the agent to a situation in which there is still a fire, and in which it has lost the capacity to follow the evacuation protocol and has not reached a safe location (state s_2); or to a state in which it has reached as a safe location (s_3). This non-determinism in π_2 captures the potential failure of the elevator as a mean for evacuation.

Defs. 3.4 and 3.5 make precise the possible plans in an LTS.

Definition 3.4. *Let R and R' be binary relations on a set S. For $S' \subseteq S$; $R(S') = \{s \in S \mid s' \in S'$ and $(s', s) \in R\}$; we write $R(s)$ instead of $R(\{s\})$. The* (sequential) composition *of R and R' is defined as $RR' = \{(s_1, s_2) \mid$ exists $s \in S$ s.t. $(s_1, s) \in R$ and $(s, s_2) \in R'\}$. In turn, let $\mathfrak{L} = \langle \mathrm{S}, \mathrm{R}, \mathrm{V}\rangle$ be an* LTS *(with* R *defined over $A \subseteq \mathrm{Act}$), and let $\pi \in \mathrm{Act}^*$ be such that $|\pi| = n$;*

$$\mathrm{R}_\pi = \begin{cases} \{(s,s) \mid s \in \mathrm{S}\} & \text{if } \pi = \epsilon \\ \emptyset & \text{if exists } 0 < k \leq n \text{ s.t. } \pi[k] \notin A \\ \mathrm{R}_{\pi[1]} \dots \mathrm{R}_{\pi[n]} & \text{otherwise.} \end{cases}$$

For $\Pi \subseteq \mathrm{Act}^*$; $\mathrm{R}_\Pi = \bigcup\{\mathrm{R}_\pi \mid \pi \in \Pi\}$.

Definition 3.5 (Strong Executability). *Let $\mathfrak{L} = \langle \mathrm{S}, \mathrm{R}, \mathrm{V}\rangle$ be an* LTS*, $s \in \mathrm{S}$, and $\pi \in \mathrm{Act}^*$; we say that π is* strongly executable (SE) *at s iff for all $0 \leq k \leq |\pi| - 1$ and all $s' \in \mathrm{R}_{\pi_k}(s)$, $\mathrm{R}_{\pi[k+1]}(s') \neq \emptyset$. The set of all states in which π is SE is $\mathrm{SE}(\pi) = \{s \in \mathrm{S} \mid \pi$ is SE at $s\}$. $\Pi \subseteq \mathrm{Act}^*$ is* strongly executable *at s iff for* every *$\pi \in \Pi$, $s \in \mathrm{SE}(\pi)$. The set of all states in which Π is SE is $\mathrm{SE}(\Pi) = \bigcap\{\mathrm{SE}(\pi) \mid \pi \in \Pi\}$.*

The notion of strong executability in Def. 3.5 states that a plan is *fail proof*, i.e., each time a plan commences at some state, it carries through. This technical requirement is inspired by conformant planning [29], and justified at a conceptual level in [32–34].

Example 3.4. *Let $\mathfrak{L}$ be the LTS in Fig. 2; the plans π_1 and π_2 in Ex. 3.2 are SE at s_1. The plan π_3 =* keep.calm *is SE everywhere. It is easy to see that the plan*

$$\pi_4 = \text{pull.alarm}; \text{use.elevator}; \text{call.999}$$

is not SE at s_1: if we take pull.alarm *and then* use.elevator*, we may land in state s_2, where it is not possible to take* call.999*.*

Thus far the picture is fairly standard. We do, however, enrich our models with a normative component.

Definition 3.6 (Normative LTS). *A* normative LTS (NLTS) *is a tuple $\mathfrak{N} = \langle \mathrm{S}, \mathrm{R}, \mathrm{V}, \mathrm{N}\rangle$ where: $\mathfrak{L} = \langle \mathrm{S}, \mathrm{R}, \mathrm{V}\rangle$ is an* LTS*, and $\mathrm{N} \subseteq \mathrm{Act}^*$ is a set of plans such that there is $\pi \in \mathrm{N}$ satisfying the condition $\mathrm{SE}(\pi) = \mathrm{S}$. We refer to* N *as the set of normative plans of $\mathfrak{N}$.*

The set N in an NLTS is intuitively understood as a set of normative plans. The requirement on N having at least one strongly executable plan can be understood as "there are norms that can always be complied with". This takes inspiration from the deontic property "ought" implies "can" (or seriality in modal logic).

Example 3.5. *Continuing with the FEEP example in Sec. 2, it would be reasonable to have: $\mathrm{N} = \{\pi_1, \pi_3, \pi_5\}$, where π_1 and π_3 are as in Exs. 3.2 and 3.4, respectively, and π_5 =* pull.alarm; use.ramp; call.999. *The set* N *captures the normative aspects of the FEEP which dictate that in case of a fire an agent shall evacuate the building using only stairs or ramps. Notice the occurrence of π_3 =* keep.calm *in* N. *This is an indication that in the case of an emergency we should always remain calm. The plan π_3 also guarantees the model is a normative LTS, as it is SE everywhere in the model (Def. 3.6).*

Our final igrendient is the perception each agent has of a given scenario. The main point to be made is that agents have their own awareness of the courses of action they can take, with some being indistinguishable. This incorporates ideas from [2].

Definition 3.7 (U-NLTS). *An* uncertainty-based normative LTSs (U-NLTS*) is a tuple $\mathfrak{M} = \langle \mathrm{S}, \mathrm{R}, \mathrm{V}, \mathrm{U}, \mathrm{N}\rangle$ where: $\mathfrak{N} = \langle \mathrm{S}, \mathrm{R}, \mathrm{V}, \mathrm{N}\rangle$ is an* NLTS*; and $\mathrm{U} : \mathrm{Agt} \to 2^{2^{\mathrm{Act}^*}}$ satisfies: $\emptyset \in \mathrm{U}(i)$, and for all $\{\Pi, \Pi'\} \subseteq \mathrm{U}(i)$, $\Pi \neq \Pi'$ implies $\Pi \cap \Pi' = \emptyset$. If $s \in \mathrm{S}$, the pair $(\mathfrak{M}, s)$ is a* pointed U-NLTS *(parentheses usually dropped).*

Intuitively, the function U in an U-NLTS indicates courses of actions that are indistinguishable from the perspective of an agent. More precisely, each $\Pi \in \mathrm{U}(i)$ captures the courses of actions that are, from the perspective of agent i, as good as any other in this set. The condition $\emptyset \in \mathrm{U}(i)$ indicates that the 'abort' plan (i.e., a plan that always fails) is possible, and that agents can tell it apart from the rest of plans. Note that $\Pi_i = \bigcup\{\Pi \mid \Pi \in \mathrm{U}(i)\}$ assigns a set of plans to an agent i. The set Π_i captures a sense of awareness for agent i, i.e., it tells what courses of action this agent may engage on. In this way, U gives rise to an equivalence relation over each Π_i, similar to the standard Epistemic Logic, but at the level of plans.

Example 3.6. *Adding to the example in Sec. 2, let us suppose that we have agents i and j, and that i has taken an occupational safety course, but j has not. Then, i should know the difference between using stairs/ramps and using the elevator to evacuate the building in case of a fire. On the other hand, for j all possible ways of exiting the building might be equally good. In this setting,*

$$\mathrm{U}(i) = \{\{\pi_1, \pi_5\}, \{\pi_4\}\} \qquad \mathrm{U}(j) = \{\{\pi_1, \pi_4, \pi_5\}\},$$

where π_1, π_4 and π_5 are as in Exs. 3.2, 3.4 and 3.5, respectively.

At this point we have all the ingredients that are necessary to introduce the formal semantics of our logic.

Definition 3.8 (Semantics). *Let* $\mathfrak{M} = \langle \mathrm{S}, \mathrm{R}, \mathrm{V}, \mathrm{U}, \mathrm{N}\rangle$ *be a* U-NLTS, $s \in \mathrm{S}$, *and* φ *be a formula;* $\mathfrak{M}, s \Vdash \varphi$ *is defined as:*

$$\begin{array}{lll}
\mathfrak{M}, s \Vdash p & \textit{iff} & p \in \mathrm{V}(s), \\
\mathfrak{M}, s \Vdash \neg\varphi & \textit{iff} & \mathfrak{M}, s \nVdash \varphi, \\
\mathfrak{M}, s \Vdash \varphi \lor \psi & \textit{iff} & \mathfrak{M}, s \Vdash \varphi \textit{ or } \mathfrak{M}, s \Vdash \psi, \\
\mathfrak{M}, s \Vdash \mathsf{N}(\psi, \varphi) & \textit{iff} & \textit{exists } \pi \in \mathrm{N} \textit{ such that} \\
& & \quad \text{(i) } [\![\psi]\!]^{\mathfrak{M}} \subseteq \mathrm{SE}(\pi) \textit{ and} \\
& & \quad \text{(ii) } \mathrm{R}_\pi([\![\psi]\!]^{\mathfrak{M}}) \subseteq [\![\varphi]\!]^{\mathfrak{M}}, \\
\mathfrak{M}, s \Vdash \mathsf{Kc}_i(\psi, \varphi) & \textit{iff} & \textit{exists } \Pi \in \mathrm{U}(i) \textit{ such that} \\
& & \quad \text{(i) } \Pi \subseteq \mathrm{N}, \\
& & \quad \text{(ii) } [\![\psi]\!]^{\mathfrak{M}} \subseteq \mathrm{SE}(\Pi), \textit{ and} \\
& & \quad \text{(iii) } \mathrm{R}_\Pi([\![\psi]\!]^{\mathfrak{M}}) \subseteq [\![\varphi]\!]^{\mathfrak{M}},
\end{array}$$

where $[\![\chi]\!]^{\mathfrak{M}} = \{s \in \mathrm{S} \mid \mathfrak{M}, s \Vdash \chi\}$.

Intuitively, $\mathsf{N}(\psi, \varphi)$ states that there is a normative plan to bring about φ given ψ (i.e., a plan that is supported by the norms in N). In turn, $\mathsf{Kc}_i(\psi, \varphi)$ states that there is a set of normative plans, all indistinguishable from the perspective of agent i, each of which brings about φ given ψ. The normative reading of the Kc_i operator is that, the agent knows how to achieve φ given ψ within the limits of some norms. Thus, we refer to Kc_i as *knowingly complying*.

Let us now turn our attention onto how to define the universal and existential modalities. Let $\mathsf{A}\varphi = \mathsf{N}(\neg\varphi, \bot)$ and $\mathsf{E}\varphi = \neg\mathsf{A}\neg\varphi$. Prop. 3.1 shows that these definitions indeed capture the usual reading of these modalities. The proof of this proposition relies on the fact that the set N is never empty and contains some π such that $\mathrm{SE}(\pi) = \mathrm{S}$, following closely the argument for the *knowing how* operator Kh in [32].

Proposition 3.1. *Let* $\mathfrak{M} = \langle \mathrm{S}, \mathrm{R}, \mathrm{V}, \mathrm{U}, \mathrm{N}\rangle$ *be a* U-NLTS, $s \in \mathrm{S}$, *and* φ *be a formula;* $\mathfrak{M}, s \Vdash \mathsf{A}\varphi$ *iff* $[\![\varphi]\!]^{\mathfrak{M}} = \mathrm{S}$.

Prop. 3.2 tells us that the modalities being considered are *global*, i.e., they hold anywhere or nowhere in the model.

Proposition 3.2. *It holds that* $\mathfrak{M}, s \Vdash \mathsf{N}(\psi, \varphi)$ *iff* $\mathfrak{M}, s \Vdash \mathsf{A}\,\mathsf{N}(\psi, \varphi)$; *and* $\mathfrak{M}, s \Vdash \mathsf{Kc}_i(\psi, \varphi)$ *iff* $\mathfrak{M}, s \Vdash \mathsf{A}\mathsf{Kc}_i(\psi, \varphi)$.

We conclude this section by summarizing our running example.

Example 3.7. *Let* $\mathfrak{M}$ *be the* U-NLTS *composed of the parts detailed in Exs. 3.3, 3.5 and 3.6; it is easy to show that:*

(1) $\mathfrak{M}, s_1 \Vdash \mathsf{A}(s \to \neg f)$ *(3)* $\mathfrak{M}, s_1 \Vdash \mathsf{N}(f \land c, s)$

(2) $\mathfrak{M}, s_1 \Vdash \mathsf{E}f$ *(4)* $\mathfrak{M}, s_1 \Vdash \mathsf{Kc}_i(f \land c, s)$

(5) $\mathfrak{M}, s_1 \nVdash \mathsf{Kc}_j(f \land c, s)$

(1) and (2) are immediate. As a witness for (3) we can take, e.g, the plan π_1 *in Ex. 3.2. As a witness for (4) we can take, e.g, the set* $\{\pi_1, \pi_5\} \in \mathrm{U}(i)$. *Failure of (5) obtains from the fact that* $\{\pi_1, \pi_4, \pi_5\} \in \mathrm{U}(j)$ *and that* $\{\pi_1, \pi_4, \pi_5\} \not\subseteq \mathrm{N}$.

4 AXIOM SYSTEM

In this section, we present a sound and complete axiom system (see Tab. 3) for DLKc. It comes to light immediately that the universal and the existential modalities A and E (definable in DLKc) are instrumental in these results.

The soundness of the axiom system $\mathcal{DLKc}$ in Tab. 3 is direct. Before establishing its completeness (Thm. 1), we offer some comments. We start with the first block of axioms. The axiomatization

Table 3: Axiom system $\mathcal{DLKc}$ for DLKc over U-NLTSs.

Axioms:	
Taut	$\vdash \varphi$ for φ a propositional tautology
DistA	$\vdash \mathsf{A}(\psi \to \varphi) \to (\mathsf{A}\psi \to \mathsf{A}\varphi)$
TA	$\vdash \mathsf{A}\varphi \to \varphi$
4KcA	$\vdash \mathsf{Kc}_i(\psi, \varphi) \to \mathsf{A}\mathsf{Kc}_i(\psi, \varphi)$
5KcA	$\vdash \neg\mathsf{Kc}_i(\psi, \varphi) \to \mathsf{A}\neg\mathsf{Kc}_i(\psi, \varphi)$
4NA	$\vdash \mathsf{N}(\psi, \varphi) \to \mathsf{A}\,\mathsf{N}(\psi, \varphi)$
5NA	$\vdash \neg\,\mathsf{N}(\psi, \varphi) \to \mathsf{A}\neg\,\mathsf{N}(\psi, \varphi)$
KcN	$\vdash \mathsf{Kc}_i(\psi, \varphi) \to \mathsf{N}(\psi, \varphi)$
DN	$\vdash \mathsf{N}(\varphi, \top)$
KcA	$\vdash (\mathsf{A}(\psi \to \chi) \land \mathsf{Kc}_i(\chi, \rho) \land \mathsf{A}(\rho \to \varphi)) \to \mathsf{Kc}_i(\psi, \varphi)$
NA	$\vdash (\mathsf{A}(\psi \to \chi) \land \mathsf{N}(\chi, \rho) \land \mathsf{A}(\rho \to \varphi)) \to \mathsf{N}(\psi, \varphi)$
Kc⊥	$\vdash \mathsf{Kc}_i(\bot, \bot)$
Rules:	
	$\dfrac{\vdash \psi \quad \vdash (\psi \to \varphi)}{\vdash \varphi}$ (MP) $\qquad \dfrac{\vdash \varphi}{\vdash \mathsf{A}\varphi}$ (Nec)

of the universal modality A needs only normality (given by axiom DistA and the rule Nec) and reflexivity (TA). As shown in [32], symmetry and transitivity for A are theorems of the system, since they can be derived from particular instances of axioms 4NA and 5NA (discussed below). The second block of axioms, 4KcA to 5NA, captures the fact that the two modalities of the language are global (see Prop. 3.2). Lastly, we turn our attention to the third block of axioms. Here, we point out that axiom KcN fixes the interaction between the two modalities, whereas DN establishes a form of *seriality* for the deontic modality N. Intuitively, DN states that, from any situation, there is always a legal way to achieve an universally valid goal ($\top$). Of the remaining axioms, KcA and NA, state that Kc_i and N, respectively, are closed under global entailment; whereas Kc⊥ tells us how agents behave in impossible situations.

Proposition 4.1. *The following formulas are derivable using the axioms and rules in Tab. 3:*

KcE $\quad (\mathsf{E}\psi \land \mathsf{Kc}_i(\psi, \varphi)) \to \mathsf{E}\varphi$; $\qquad$ NE $\quad (\mathsf{E}\psi \land \mathsf{N}(\psi, \varphi)) \to \mathsf{E}\varphi$

At this point, we turn our attention to completeness. We begin with some preliminary definitions.

Definition 4.1. *Let* $\mathbf{\Phi}$ *be the set of all* maximally consistent sets (MCS) *of formulas (w.r.t.* $\mathcal{DLKc}$*); for any* $\Gamma \in \mathbf{\Phi}$, *define:*

$$\Gamma|_{\mathsf{N}} = \{\mathsf{N}(\psi, \varphi) \mid \mathsf{N}(\psi, \varphi) \in \Gamma\} \qquad \Gamma|_{\mathsf{A}} = \{\mathsf{A}\varphi \mid \mathsf{A}\varphi \in \Gamma\}$$

$$\Gamma|_{\mathsf{Kc}_i} = \{\mathsf{Kc}_i(\psi, \varphi) \mid \mathsf{Kc}_i(\psi, \varphi) \in \Gamma\} \qquad \Gamma|_{\mathsf{Kc}} = \bigcup\{\Gamma|_{\mathsf{Kc}_i} \mid i \in \mathrm{Agt}\}.$$

For $\Gamma \in \mathbf{\Phi}$; *define* $\mathrm{Act}^\Gamma = \{\langle \psi, \varphi\rangle \mid \mathsf{N}(\psi, \varphi) \in \Gamma\}$.

Remark 4.1. *Note that* Act^Γ *is denumerable, thus it is an adequate set of actions for building a model. Note also that* Act^Γ *fixes a new signature. This causes no problem since the operators of the language cannot see the names of the actions; i.e., we can define a mapping from* Act^Γ *to any particular* Act *and preserve the original signature.*

Let us take the first step towards an adequate notion of canonical model $\mathfrak{M}^\Gamma$ for an MCS $\Gamma \in \mathbf{\Phi}$. Using standard ideas from modal logic [7], we would take the set S of states of $\mathfrak{M}^\Gamma$ to be $\mathbf{\Phi}$. In doing this, however, we run into a problem. We may have, $\mathsf{Kc}_i(\psi, \varphi) \in s$ and $\neg\mathsf{Kc}_i(\psi, \varphi) \in s'$, for some $\{s, s'\} \subseteq \mathrm{S}$ (since $\mathrm{S} = \mathbf{\Phi}$ contains all MCSs w.r.t. $\mathcal{DLKc}$). This causes the *Truth-Lemma* to fail: it should happen that $\mathfrak{M}^\Gamma, s \Vdash \mathsf{Kc}_i(\psi, \varphi)$ iff for all $s'' \in \mathrm{S}$, $\mathfrak{M}^\Gamma, s'' \Vdash \mathsf{Kc}_i(\psi, \varphi)$ (see Prop. 3.2). Yet, we have s' such that $\mathfrak{M}^\Gamma, s' \Vdash \neg\mathsf{Kc}_i(\psi, \varphi)$, and so

$\mathfrak{M}^\Gamma, s' \not\Vdash \mathsf{Kc}_i(\psi, \varphi)$, a contradiction. Moreover, a similar argument can be puth forth for N.

The scenario described above tells us that we need to do some extra work in building our canonical model, similar to what happens in e.g., [17] and in [18] for the universal modality and Propositional Dynamic Logic (PDL), respectively. In those cases, the corresponding structure needs to satisfy some additional constraint. In the former, the situation is similar than ours, as the canonical model is generated by the modality A. For the latter, the canonical model is filtrated in order to characterize the transitive closure of a relation.

Simply put, we need to select the appropriate set of MCS in order to fulfill the 'globality' requirement for our modal formulas, established in the block of axioms 4KcA to 5NA. This extra work is made precise in Def. 4.2.

Definition 4.2. *The* canonical model *of an MCS of formulas* $\Gamma \in \Phi$ *is a tuple* $\mathfrak{M}^\Gamma_c = \langle S^\Gamma, R^\Gamma, V^\Gamma, U^\Gamma, N^\Gamma \rangle$ *where:*

$$S^\Gamma = \{\Delta \in \Phi \mid \Delta|_A = \Gamma|_A\}$$
$$R^\Gamma_{\langle\psi,\varphi\rangle} = \{(\Delta_1, \Delta_2) \in S^\Gamma \times S^\Gamma \mid \mathsf{N}(\psi, \varphi) \in \Gamma, \psi \in \Delta_1, \varphi \in \Delta_2\}$$
$$R^\Gamma = \{R^\Gamma_{\langle\psi,\varphi\rangle} \mid \mathsf{N}(\psi, \varphi) \in \Gamma\}$$
$$V^\Gamma(\Delta) = \{p \in \mathsf{Prop} \mid p \in \Delta\}$$
$$U^\Gamma(i) = \{\{\langle\psi, \varphi\rangle\} \mid \mathsf{Kc}_i(\psi, \varphi) \in \Gamma\} \cup \{\emptyset\}$$
$$N^\Gamma = \{\langle\psi, \varphi\rangle \mid \mathsf{N}(\psi, \varphi) \in \Gamma\}.$$

Notice that $\mathfrak{M}^\Gamma_c$ is generated by formulas of the form $\mathsf{A}\varphi$; i.e., the (global) formulas that occur in Γ.

Proposition 4.2. *The following are immediate for* $\mathfrak{M}^\Gamma_c$*:*
(1) $\{\langle\psi, \varphi\rangle\} \in U^\Gamma(i)$ *implies* $\langle\psi, \varphi\rangle \in N^\Gamma$*;*
(2) $\langle\psi, \varphi\rangle \in N^\Gamma$ *iff* $\langle\psi, \varphi\rangle \in \mathsf{Act}^\Gamma$*;*
(3) $R^\Gamma_{\langle\psi,\varphi\rangle} \neq \emptyset$ *implies* $\mathsf{N}(\psi, \varphi) \in \Gamma$.

Items (1) of Prop. 4.2 follows by axiom KcN. Items (2) and (3) follow by definition of $\mathfrak{M}^\Gamma_c$. Now, we need to show that $\mathfrak{M}^\Gamma_c$ is a proper U-NLTS.

Proposition 4.3. $\mathfrak{M}^\Gamma_c = \langle S^\Gamma, R^\Gamma, V^\Gamma, U^\Gamma, N^\Gamma \rangle$ *is a* U-NLTS.

Proof. It is clear that $\langle S^\Gamma, R^\Gamma, V^\Gamma \rangle$ is an LTS (e.g., $S^\Gamma \neq \emptyset$, as $\Gamma \in S^\Gamma$). Then, we need to show that there exists $\pi \in N^\Gamma$ s.t. $\mathsf{SE}(\pi) = S^\Gamma$, as per Def. 3.6. Notice that $\mathsf{N}(\varphi, \top) \in \Gamma$ for every $\varphi \in \mathsf{DLKc}$ (by DN). In particular, $\mathsf{N}(\top, \top) \in \Gamma$. Hence, $\langle\top, \top\rangle \in N^\Gamma$, and $\mathsf{SE}(\langle\top, \top\rangle) = S^\Gamma$.

It remains to show that U^Γ satisfies the conditions of Def. 3.7. By definition, $\emptyset \in U^\Gamma(i)$. Let $\Pi_1, \Pi_2 \in U^\Gamma_i - \{\emptyset\}$ be s.t. $\Pi_1 = \{\langle\psi_1, \varphi_1\rangle\} \neq \{\langle\psi_2, \varphi_2\rangle\} = \Pi_2$ (recall that Π_1, Π_2 are singletons). Then, $\Pi_1 \cap \Pi_2 = \emptyset$. □

Below we state some properties of $\mathfrak{M}^\Gamma_c$, that will be useful in what follows. We start with a property about the global relatios between the states of $\mathfrak{M}^\Gamma_c$, whose proof relies on the axioms of the block 4KcA to 5NA.

Proposition 4.4. *Let* $\{\Delta_1, \Delta_2\} \subseteq S^\Gamma$ *and* $\mathsf{X} \in \{\mathsf{Kc}_i, \mathsf{N}\}$*; we have* $\Delta_1|_\mathsf{X} = \Delta_2|_\mathsf{X} = \Gamma|_\mathsf{X}$.

Next, we establish some properties about the structure of $\mathfrak{M}^\Gamma_c$ (see [2, 34] for details).

Proposition 4.5. *If* $R^\Gamma_{\langle\psi,\varphi\rangle}(\Delta) \neq \emptyset$*, then, for all* $\Delta' \in S^\Gamma$*,* $\varphi \in \Delta'$ *implies* $\Delta' \in R^\Gamma_{\langle\psi,\varphi\rangle}(\Delta)$.

Proposition 4.6. *For any formula* φ*; if* $\varphi \in \Delta$ *for every* $\Delta \in S^\Gamma$*, then* $\mathsf{A}\varphi \in \Delta$ *for every* $\Delta \in S^\Gamma$.

Proposition 4.7. *If* $\psi \in \Delta$ *then,* $R^\Gamma_{\langle\psi',\varphi'\rangle}(\Delta) \neq \emptyset$*, implies for all* $\Delta' \in S^\Gamma$*,* $\mathsf{A}(\psi \to \psi') \in \Delta'$.

Proposition 4.8. *Let* $\mathsf{X} \in \{\mathsf{Kc}_i, \mathsf{N}\}$*; if there is* $\Theta \in S^\Gamma$ *such that* $\{\psi, \mathsf{X}(\psi, \varphi)\} \subseteq \Theta$*, then there is* $\Theta' \in S^\Gamma$ *such that* $\varphi \in \Theta'$.

Props. 4.4 to 4.8 are instrumental in our proof of the Truth Lemma for $\mathfrak{M}^\Gamma_c$, which is stated below.

Lemma 4.1 (Truth lemma). *Let* $\Gamma \in \Phi$*, and let* $\mathfrak{M}^\Gamma_c$ *be as in Def. 4.2; for all* $\Delta \in S^\Gamma$*, and for all* $\varphi \in \mathsf{DLKc}$*,* $\mathfrak{M}^\Gamma_c, \Delta \Vdash \varphi$ *iff* $\varphi \in \Delta$.

Proof. Let $\mathfrak{M}^\Gamma_c = \langle S^\Gamma, R^\Gamma, V^\Gamma, U^\Gamma, N^\Gamma \rangle$. The proof is by induction on the structure of φ. The atomic and Boolean cases are as usual, so we focus on the Kc_i case (the case of N being similar).

Case $\varphi = \mathsf{Kc}_i(\psi, \rho)$**:**. ($\Rightarrow$) Suppose $\mathfrak{M}^\Gamma_c, \Theta \Vdash \mathsf{Kc}_i(\psi, \rho)$. Then, there is $\Pi \in U^\Gamma(i)$ such that: $\Pi \subseteq N^\Gamma$, $[\![\psi]\!]^{\mathfrak{M}^\Gamma_c} \subseteq \mathsf{SE}(\Pi)$ and $R_\Pi([\![\psi]\!]^{\mathfrak{M}^\Gamma_c}) \subseteq [\![\rho]\!]^{\mathfrak{M}^\Gamma_c}$. We have two cases:

- If $[\![\psi]\!]^{\mathfrak{M}^\Gamma_c} = \emptyset$, then by IH, $\neg\psi \in \Theta'$ for all $\Theta' \in S^\Gamma$. Thus, by Prop. 4.6, $\mathsf{A}\neg\psi \in \Theta'$ for all $\Theta' \in S^\Gamma$, and therefore, $\mathsf{A}(\psi \to \bot) \in \Theta'$. Since $\bot \to \rho$ is a tautology, by Nec, $\mathsf{A}(\bot \to \rho) \in \Theta'$. Using an instance of KcA, $(\mathsf{A}(\psi \to \bot) \wedge \mathsf{Kc}_i(\bot, \bot) \wedge \mathsf{A}(\bot \to \rho)) \to \mathsf{Kc}_i(\psi, \rho) \in \Theta'$. Then, by Kc$\bot$ and MP, we get that $\mathsf{Kc}_i(\psi, \rho) \in \Theta'$, for all $\Theta' \in S^\Gamma$. Thus, $\mathsf{Kc}_i(\psi, \rho) \in \Theta$.
- If $[\![\psi]\!]^{\mathfrak{M}^\Gamma_c} \neq \emptyset$, then $\Pi = \{\langle\psi', \varphi'\rangle\}$ since $\Pi = \emptyset$ would force $[\![\psi]\!]^{\mathfrak{M}^\Gamma_c} = \emptyset$. Take $\Theta \in S^\Gamma$ such that $\mathfrak{M}^\Gamma_c, \Theta, \Vdash \psi$. By IH, $\psi \in \Theta$. Moreover, for all $\Theta' \in S^\Gamma$, if $\psi \in \Theta'$ then:
  - Θ' has an $R^\Gamma_{\langle\psi',\varphi'\rangle}$-successor, and
  - for all $\Theta'' \in S^\Gamma$ s.t. $(\Theta', \Theta'') \in R^\Gamma_{\langle\psi',\varphi'\rangle}$, $\rho \in \Theta''$ (IH).

  Note that as Θ' has an $R^\Gamma_{\langle\psi',\varphi'\rangle}$-successor, $\psi' \in \Theta$, and therefore, $\psi \to \psi'$, for all $\Theta' \in S^\Gamma$. By Prop. 4.7, $\mathsf{A}(\psi \to \psi') \in \Theta'$ for all $\Theta' \in S^\Gamma$. By Prop. 4.5, every $\Theta'' \in S^\Gamma$ such that $\varphi' \in \Theta''$ can be $R^\Gamma_{\langle\psi',\varphi'\rangle}$-reached from Θ'. Therefore, for every $\Theta' \in S^\Gamma$ such that $\psi \in \Theta'$ we have that every $\Theta'' \in S^\Gamma$ such that $\varphi' \in \Theta''$ can be $R^\Gamma_{\langle\psi',\varphi'\rangle}$-reached from Θ'; i.e., $\Theta'' \in [\![\varphi']\!]^{\mathfrak{M}^\Gamma_c} \subseteq R^\Gamma_\Pi([\![\psi]\!]^{\mathfrak{M}^\Gamma_c}) \subseteq [\![\rho]\!]^{\mathfrak{M}^\Gamma_c}$. Then, we get $\rho \in \Theta''$. Thus, for all $\Theta'' \in S^\Gamma$, $\varphi' \to \rho \in \Theta''$. Using Prop. 4.6, for all $\Theta'' \in S^\Gamma$, $\mathsf{A}(\varphi' \to \rho) \in \Theta''$. Finally, putting all together, for all $\Theta' \in S^\Gamma$, $\{\mathsf{A}(\psi \to \psi'), \mathsf{Kc}_i(\psi', \varphi'), \mathsf{A}(\varphi' \to \rho)\} \subset \Theta'$. By axiom KcA, $\mathsf{Kc}_i(\psi, \rho) \in \Theta'$ and thus, $\mathsf{Kc}_i(\psi, \rho) \in \Theta$.

($\Leftarrow$) Suppose $\mathsf{Kc}_i(\psi, \rho) \in \Theta$. Then, by Prop. 4.4, $\mathsf{Kc}_i(\psi, \rho) \in \Theta'$ for all $\Theta' \in S^\Gamma$. Moreover, $\mathsf{Kc}_i(\psi, \rho) \in \Gamma$ and $R^\Gamma_{\langle\psi,\rho\rangle}$ is defined. To prove that $\mathfrak{M}^\Gamma_c, \Theta \Vdash \mathsf{Kc}_i(\psi, \rho)$, we have to consider two cases:

- There is no Θ' such that $\psi \in \Theta'$. By IH, $[\![\psi]\!]^{\mathfrak{M}^\Gamma_c} = \emptyset$. Using $\Pi = \emptyset$, we trivially have that $\mathfrak{M}^\Gamma_c, \Theta \Vdash \mathsf{Kc}_i(\psi, \rho)$.
- There is Θ' such that $\psi \in \Theta'$: by Prop. 4.8, there is Θ'' such that $\rho \in \Theta''$. By IH, $\mathfrak{M}^\Gamma_c, \Theta' \Vdash \psi$ and $\mathfrak{M}^\Gamma_c, \Theta'' \Vdash \rho$. Since it is defined, $\Pi = \{\langle\psi, \rho\rangle\}$ is SE at all ψ-states (since there is an

$R^{\Gamma}_{\langle\psi,\rho\rangle}$-successor Θ''), reaches from these only ρ-states via Π (by construction of $R^{\Gamma}_{\langle\psi,\rho\rangle}$), and $\{\langle\psi,\rho\rangle\} \subseteq N^{\Gamma}$ (by Prop. 4.2). Thus, $\mathfrak{M}^{\Gamma}_c, \Theta \Vdash \mathsf{Kc}_i(\psi,\rho)$.

□

Following from Lemma 4.1, and using a standard argument, we establish the following result.

Theorem 1. *The axiom system $\mathcal{DLKc}$ in Tab. 3 is sound and strongly complete for* DLKc *over the class of all* U-NLTS*s.*

5 COMPLEXITY

In this section, we investigate the computational complexity of the satisfiability problem of DLKc. For this logic, we will establish membership in NP by showing a polynomial size model property.

Given a formula, we will show that it is possible to select just a piece of the canonical model which is relevant for its evaluation. The selected model will preserve satisfiability, and moreover, its size will be polynomial w.r.t. the size of the input formula.

Definition 5.1 (Selection function). *Let Γ be a MCS, and let $\mathfrak{M}^{\Gamma}_c = \langle S^{\Gamma}, R^{\Gamma}, V^{\Gamma}, U^{\Gamma}, N^{\Gamma}\rangle$ be a canonical model; let $w \in S^{\Gamma}$ and φ be a formula. Define $\mathsf{Act}_{\varphi} = \{\langle\theta_1,\theta_2\rangle \in \mathsf{Act}^{\Gamma} \mid \mathsf{X}(\theta_1,\theta_2) \in sf(\varphi)\} \cup \{\langle\top,\top\rangle\}$, with $\mathsf{X} \in \{\mathsf{Kc}_i, \mathsf{N}\}$ and $sf(\varphi)$ the set of subformulas of φ defined in the usual way. A* canonical selection function sel^{φ}_w *is a function that takes $\mathfrak{M}^{\Gamma}_c$, w and φ as input, returns a set $S' \subseteq S^{\Gamma}$, and is such that:*

(1) $\mathsf{sel}^{\varphi}_w(p) = \{w\}$;
(2) $\mathsf{sel}^{\varphi}_w(\neg\varphi_1) = \mathsf{sel}^{\varphi}_w(\varphi_1)$
(3) $\mathsf{sel}^{\varphi}_w(\varphi_1 \vee \varphi_2) = \mathsf{sel}^{\varphi}_w(\varphi_1) \cup \mathsf{sel}^{\varphi}_w(\varphi_2)$;
(4) *If* $[\![\mathsf{X}(\varphi_1,\varphi_2)]\!]^{\mathfrak{M}^{\Gamma}_c} \neq \emptyset$ *and* $[\![\varphi_1]\!]^{\mathfrak{M}^{\Gamma}_c} = \emptyset$ *for* $\mathsf{X} \in \{\mathsf{Kc}_i, \mathsf{N}\}$*:*
 $\mathsf{sel}^{\varphi}_w(\mathsf{X}(\varphi_1,\varphi_2)) = \{w\}$;
(5) *If* $[\![\mathsf{X}(\varphi_1,\varphi_2)]\!]^{\mathfrak{M}^{\Gamma}_c} \neq \emptyset$ *and* $[\![\varphi_1]\!]^{\mathfrak{M}^{\Gamma}_c} \neq \emptyset$ *for* $\mathsf{X} \in \{\mathsf{Kc}_i, \mathsf{N}\}$*:*
 $\mathsf{sel}^{\varphi}_w(\mathsf{X}(\varphi_1,\varphi_2)) = \{w_1, w_2\} \cup \mathsf{sel}^{\varphi}_{w_1}(\varphi_1) \cup \mathsf{sel}^{\varphi}_{w_2}(\varphi_2)$,
 where w_1, w_2 are s.t. $(w_1, w_2) \in R^{\Gamma}_{\langle\varphi_1,\varphi_2\rangle}$;
(6) *If* $[\![\mathsf{Kc}_i(\varphi_1,\varphi_2)]\!]^{\mathfrak{M}^{\Gamma}_c} = \emptyset$ *(note that* $[\![\varphi_1]\!]^{\mathfrak{M}^{\Gamma}_c} \neq \emptyset$*):*
 For each $\{\langle\psi_1,\psi_2\rangle\} = \{a\} \in U^{\Gamma}(i) \cap \mathsf{Act}_{\varphi}$*:*
 (a) *if* $[\![\varphi_1]\!]^{\mathfrak{M}^{\Gamma}_c} \not\subseteq SE(a)$*:*
  add $\mathsf{sel}^{\varphi}_{w_1}(\varphi_1) \cup \{w_1\}$ *to* $\mathsf{sel}^{\varphi}_w(\mathsf{Kc}_i(\varphi_1,\varphi_2))$,
  where $w_1 \in [\![\varphi_1]\!]^{\mathfrak{M}^{\Gamma}_c}$ *and* $w_1 \notin SE(a)$;
 (b) *if* $R^{\Gamma}_a([\![\varphi_1]\!]^{\mathfrak{M}^{\Gamma}_c}) \not\subseteq [\![\varphi_2]\!]^{\mathfrak{M}^{\Gamma}_c}$*:*
  add $\{w_1,w_2\} \cup \mathsf{sel}^{\varphi}_{w_1}(\varphi_1) \cup \mathsf{sel}^{\varphi}_{w_2}(\varphi_2)$ *to* $\mathsf{sel}^{\varphi}_w(\mathsf{Kc}_i(\varphi_1,\varphi_2))$,
  where $w_1 \in [\![\varphi_1]\!]^{\mathfrak{M}^{\Gamma}_c}$, $w_2 \in R^{\Gamma}_a(w_1)$ *and* $w_2 \notin [\![\varphi_2]\!]^{\mathfrak{M}^{\Gamma}_c}$;
(7) *If* $[\![\mathsf{N}(\varphi_1,\varphi_2)]\!]^{\mathfrak{M}^{\Gamma}_c} = \emptyset$ *(note that* $[\![\varphi_1]\!]^{\mathfrak{M}^{\Gamma}_c} \neq \emptyset$*):*
 For each $\langle\psi_1,\psi_2\rangle = a \in N^{\Gamma} \cap \mathsf{Act}_{\varphi}$*:*
 (a) *if* $[\![\varphi_1]\!]^{\mathfrak{M}^{\Gamma}_c} \not\subseteq SE(a)$*:*
  add $\{w_1\} \cup \mathsf{sel}^{\varphi}_{w_1}(\varphi_1)$ *to* $\mathsf{sel}^{\varphi}_w(\mathsf{N}(\varphi_1,\varphi_2))$,
  where $w_1 \in [\![\varphi_1]\!]^{\mathfrak{M}^{\Gamma}_c}$ *and* $w_1 \notin SE(a)$;
 (b) *if* $R^{\Gamma}_a([\![\varphi_1]\!]^{\mathfrak{M}^{\Gamma}_c}) \not\subseteq [\![\varphi_2]\!]^{\mathfrak{M}^{\Gamma}_c}$*:*
  add $\{w_1,w_2\} \cup \mathsf{sel}^{\varphi}_{w_1}(\varphi_1) \cup \mathsf{sel}^{\varphi}_{w_2}(\varphi_2)$ *to* $\mathsf{sel}^{\varphi}_w(\mathsf{N}(\varphi_1,\varphi_2))$,
  where $w_1 \in [\![\varphi_1]\!]^{\mathfrak{M}^{\Gamma}_c}$, $w_2 \in R^{\Gamma}_a(w_1)$ *and* $w_2 \notin [\![\varphi_2]\!]^{\mathfrak{M}^{\Gamma}_c}$.

The overall idea is inspired by selection methods in modal logics [8], and in knowing how logic [2]. The selection function picks enough states of the canonical model, in order to ensure the preservation of the truth of the subformulas of a given input formula. In addition, we also need to pick the proper set of normed plans and uncertainty sets. This is made precise in the following definition.

Definition 5.2 (Selected model). *Let $\mathfrak{M}^{\Gamma}_c$ be the canonical model for an MCS Γ, w a state in $\mathfrak{M}^{\Gamma}_c$, and φ a formula. Let sel^{φ}_w be a selection function, we define the* model selected by sel^{φ}_w *as $\mathfrak{M}^{\varphi}_w = \langle S^{\varphi}_w, R^{\varphi}_w, V^{\varphi}_w, U^{\varphi}_w, N^{\varphi}_w\rangle$, where*

$S^{\varphi}_w = \mathsf{sel}^{\varphi}_w(\varphi)$;

$(R^{\varphi}_w)_{\langle\theta_1,\theta_2\rangle} = R^{\Gamma}_{\langle\theta_1,\theta_2\rangle} \cap (S^{\varphi}_w \times S^{\varphi}_w)$, *for each* $\langle\theta_1,\theta_2\rangle \in \mathsf{Act}_{\varphi}$;

$N^{\varphi}_w = N^{\Gamma} \cap \mathsf{Act}_{\varphi}$;

$(U^{\varphi}_w)(i) = \{\{a\} \in U^{\Gamma}(i) \mid a \in \mathsf{Act}_{\varphi}\} \cup \{\emptyset\}$, *for* $i \in \mathsf{Agt}$;

V^{φ}_w *is the restriction of* V^{Γ} *to* S^{φ}_w.

Proposition 5.1. *$\mathfrak{M}^{\varphi}_w = \langle S^{\varphi}_w, R^{\varphi}_w, V^{\varphi}_w, U^{\varphi}_w, N^{\varphi}_w\rangle$ is a* U-NLTS*. Moreover, $\Pi \in U^{\varphi}_w(i)$ implies $\Pi \subseteq N^{\varphi}_w$.*

Proof. The structure $\langle S^{\varphi}_w, R^{\varphi}_w, V^{\varphi}_w\rangle$ is an LTS as $\mathsf{sel}^{\varphi}_w(\varphi) \neq \emptyset$. Since we preserved $\langle\top,\top\rangle$ in N^{φ}_w, and (U^{φ}_w) is just a restriction of U^{Γ} to Act_{φ}, it is easy to see that it meets the conditions of Def. 3.7. Thus, $\mathfrak{M}^{\varphi}_w$ is a U-NLTS. The last implication follows by definition. □

Below we state the crucial property for characterizing the complexity of checking the satisfiability of DLKc-formulas.

Proposition 5.2. *Let $\mathfrak{M}^{\Gamma}_c$ be a canonical model, w a state in $\mathfrak{M}^{\Gamma}_c$ and φ a formula. Let $\mathfrak{M}^{\varphi}_w$ be the selected model by a selection function sel^{φ}_w. Then, $\mathfrak{M}^{\Gamma}_c, w \Vdash \varphi$ implies that for all ψ subformula of φ, and for all $v \in S^{\varphi}_w$, we have that $\mathfrak{M}^{\Gamma}_c, v \Vdash \psi$ iff $\mathfrak{M}^{\varphi}_w, v \Vdash \psi$. Moreover, $\mathfrak{M}^{\varphi}_w$ is polynomial on the size of φ.*

Proof (Sketch). The proof of that $\mathfrak{M}^{\varphi}_w$ preserves the satisfiability of formulas follows by a standard induction in the size of φ. It remains to show that $\mathfrak{M}^{\varphi}_w$ is polynomial on the size of φ. The selection function adds states from $\mathfrak{M}^{\Gamma}_c$, only for each subformula of φ with Kc_i or N as the outermost connective. The number of states added at each time is polynomial in the size of φ. Hence, the size of S^{φ}_w is polynomial. Since $(U^{\varphi}_w)(i)$ and N^{φ}_w are also polynomial, the size of $\mathfrak{M}^{\varphi}_w$ is polynomial in the size of φ. □

In order to prove that the satisfiability problem is in NP, it remains to show that the model checking problem is in P (the proof is omitted due to lack of space, but it is similar to the one provided in [2] for an epistemic knowing how logic).

Proposition 5.3. *The model checking problem for* DLKc *is in* P.

Now, we are in position to characterize the complexity of the satisfiability problem for DLKc.

Theorem 2. *The satisfiability problem for* DLKc *is* NP*-complete.*

Proof. Hardness follows from NP-completeness of propositional logic (a fragment of DLKc). By Prop. 5.2, each satisfiable formula φ has a model of polynomial size on φ. Thus, we can guess a polymonial model $\mathfrak{M}, w$, and test $\mathfrak{M}, w \Vdash \varphi$ (which can be done in polynomial time, due to Prop. 5.3). Therefore, the result follows. □

6 REASONING ABOUT ABILITIES

So far, we studied a logical formalism that enables us to express and reason about normative courses of actions and responsibilities for agents. These notions are syntactically captured in our logic using the modalities N and Kc_i, respectively. From a semantic standpoint, these modalities are interpreted on a model relative to a set N of 'normed' plans. This raises the question of whether it is possible to reason in our logic about the possible plans the agents may engage on, i.e., expressing what an agent 'can do', independently of the given norms or even of their own individual perception. Having this new feature is useful for studying interactions between the given abilities and what agents can do according to the norms. To this end, in this section, we investigate the impact of adding a new modality (called S) to DLKc and refer to the resulting logic as DLKc^+.

Definition 6.1. *The language of* DLKc^+ *is defined by:*

$$\varphi, \psi ::= p \mid \neg\varphi \mid \varphi \vee \psi \mid \mathsf{S}(\psi, \varphi) \mid \mathsf{N}(\psi, \varphi) \mid \mathsf{Kc}_i(\psi, \varphi),$$

where: $p \in \mathsf{Prop}$ *and* $i \in \mathsf{Agt}$. *Intuitively, a formula* $\mathsf{S}(\psi, \varphi)$ *is read as:* "there is a course of action that brings about φ given ψ".

The semantic clause for the modality S is given below. It is worth noticing that this semantic clause is exactly the one for the knowing how modality of [32, 33]. As argued in [2], such a modality can be seen as an ability modality, rather than an epistemic one. We adhere to [2] and take S as an ability modality.

Definition 6.2. *Let* $\mathfrak{M} = \langle S, R, V, U, N\rangle$ *be a* U-NLTS, $s \in S$, *and* φ *be a formula; Def. 3.8 is extended to account for the operator* S *as:*

$$\mathfrak{M}, s \Vdash \mathsf{S}(\psi, \varphi) \quad \textit{iff} \quad \text{exists } \pi \in \mathsf{Act}^* \text{ such that (i) } [\![\psi]\!]^{\mathfrak{M}} \subseteq \mathrm{SE}(\pi) \text{ and (ii) } \mathrm{R}_\pi([\![\psi]\!]^{\mathfrak{M}}) \subseteq [\![\varphi]\!]^{\mathfrak{M}}.$$

The following example motivates the use of S.

Example 6.1. *In the context of our FEEP example, the formula* $\mathsf{S}(\top, s)$ *expresses that 'there is a course of action that always* ($\top$) *allows agents to reach a safe location', whereas* $\mathsf{S}(f \wedge c, s)$ *states that 'there is a course of action that always leads to a safe location in case of a fire event, provided there is the capacity (c) of doing so'. In the context of the* LTS *in Fig. 2, we have that*

(1) $\mathfrak{M}, s_1 \Vdash \mathsf{S}(f \wedge c, s)$ *(2)* $\mathfrak{M}, s_1 \nVdash \mathsf{S}(\top, s)$.

For (1), we can take the plan $\pi_4 = \mathsf{pull.alarm}; \mathsf{use.ramp}; \mathsf{call.999}$ *as a witness. Notice that (2) holds, since in state* s_2, *there is no plan for reaching a safe location (i.e., a state in which s holds). Intuitively, the* LTS *in Fig. 2 can be seen as illustrating the actions available in the case of a fire in a building. Item (1) tells us that in such a scenario it is possible to reach a safe location if there is the capacity to do so; whereas item (2) tells us that it is not always possible to reach a safe location, e.g., if we are trapped in an elevator.*

An axiomatization for DLKc^+ is obtained by adding the axiom schemas in Tab. 4 to the axiom system introduced in Tab. 3. Notice that, NS establishes the interaction between N and S. Intuitively, it tells us that whichever is regulated by norms is also feasible. This rules out normative systems in which certain norms are impossible –e.g., if we think of norms in terms of obligations, this axiom tells us that our logic adheres to the principle *impossibilium nulla obligatio est*, which states that impossible norms shall be excused. As one

Table 4: Additional axioms for DLKc^+.

4SA	$\vdash \mathsf{S}(\psi, \varphi) \to \mathsf{AS}(\psi, \varphi)$	NS	$\vdash \mathsf{N}(\psi, \varphi) \to \mathsf{S}(\psi, \varphi)$
5SA	$\vdash \neg\mathsf{S}(\psi, \varphi) \to \mathsf{A}\neg\mathsf{S}(\psi, \varphi)$	EmpS	$\vdash \mathsf{A}(\psi \to \varphi) \to \mathsf{S}(\psi, \varphi)$
		CompS	$\vdash (\mathsf{S}(\psi, \chi) \wedge \mathsf{S}(\chi, \varphi)) \to \mathsf{S}(\psi, \varphi)$

would expect, the converse of NS is not a theorem of the logic –i.e., it is possible for certain courses of actions not to be regulated by norms. Lastly, the axioms EmpS and CompS capture some intuitions behind the possible courses of actions. These axioms are present in the original proposal for a knowing how modality [32]. In particular, EmpS tells us that it is possible to turn universally valid implications into abilities by doing nothing (witnessed by the empty plan ϵ); whereas CompS tells us that courses of actions that have a common goal/context can be composed. In [2], it is argued that they account for some level of omniscience that one might disagree with. This challenge, however, is set aside here since S is taken not as an epistemic modality but as an *ability* modality.

Theorem 3. *The axioms and rules in Tabs. 3 and 4 yield a sound and strongly complete axiom system for* DLKc^+ *over* U-NLTS*s.*

We conclude with a comment on the computational behaviour of DLKc^+. The semantics of S is exactly the one of Kh in [32, 33]. It is shown in [24], that the satisfiability problem is decidable for this logic (in fact, the result is proved for a more general logic). We claim that following ideas from [26], we can provide a complexity characterization for this logic, whose status is still an open problem.

7 FINAL REMARKS

We presented a deontic logic for modelling the notion of knowingly complying with a given set of norms. The logic features two modalities N and Kc_i, one refining the other. On the one hand, N models those abilities that are normed (i.e., those that are within the limits of the law). On the other hand, Kc_i is the modality of 'knowingly complying' (i.e., it models the conciousness of an agent when achieving a certain goal by using a normed course of action). We used a fire emergency evauation plan (FEEP) as a running example to illustrate the components of the logic. Then, we introduced a sound and strongly complete axiom system to provide an account on how the modalities interact in the logic. Moreover, we showed that the satisfiability problem for the logic is NP-complete, relying on a small model property. Finally, we extended the logic with a modality S to capture the general abilities of the agents. We studied the effects of including such a modality. This result illustrates the flexibility of our framework, and shows how the three aspects in our deontic systems interact among each other.

There are several interesting directions for future work. First, it would be interesting to characterize the exact complexity of the satisfiability poblem of the extended logic DLKc^+. This would give us, as a by product, the exact complexity of the basic knowing how logic from [32, 34]. Second, by playing with the relation between the set of plans $\mathrm{U}(i)$ of each agent and the set of norms N, it is possible to establish different levels of responsibility for the agents. This relates with the different notions of *knowingly doing* of [10–12]. Finally, our approach enables us to impose new restrictions on the different components of the model (or weakening them), and obtain new logics. It would be interesting to study these different systems in a deontic context.

8 TECHNICAL APPENDIX

8.1 Proofs of Sec. 4

Proof. (Prop. 4.1)

(KcE). By Taut, it is equivalent to prove $\vdash (\mathsf{E}\psi \wedge \mathsf{Kc}_i(\psi,\varphi) \wedge \mathsf{A}\neg\varphi) \rightarrow \bot$. By using KcN, we have that $\vdash (\mathsf{E}\psi \wedge \mathsf{Kc}_i(\psi,\varphi) \wedge \mathsf{A}\neg\varphi) \rightarrow (\mathsf{E}\psi \wedge \mathsf{O}(\psi,\varphi) \wedge \mathsf{S}(\varphi,\bot))$ (notice we unfolded the definition of A). Similarly, using NS, $\vdash (\mathsf{E}\psi \wedge \mathsf{O}(\psi,\varphi) \wedge \mathsf{S}(\varphi,\bot)) \rightarrow (\mathsf{E}\psi \wedge \mathsf{S}(\psi,\varphi) \wedge \mathsf{S}(\varphi,\bot))$. Thus, by CompS, $\vdash (\mathsf{E}\psi \wedge \mathsf{S}(\psi,\varphi) \wedge \mathsf{S}(\varphi,\bot)) \rightarrow (\mathsf{E}\psi \wedge \mathsf{S}(\psi,\bot))$. By A's definition and Taut ($\varphi \rightarrow \varphi$), $\vdash (\mathsf{E}\psi \wedge \mathsf{S}(\psi,\bot)) \rightarrow \bot$. Thus, by the transitivity of the implication, $\vdash (\mathsf{E}\psi \wedge \mathsf{Kc}_i(\psi,\varphi) \wedge \mathsf{A}\neg\varphi) \rightarrow \bot$.

(NE). Using in this case NS only, we have that, $\vdash (\mathsf{E}\psi \wedge \mathsf{O}(\psi,\varphi) \wedge \mathsf{S}(\varphi,\bot)) \rightarrow (\mathsf{E}\psi \wedge \mathsf{S}(\psi,\varphi) \wedge \mathsf{S}(\varphi,\bot))$. The rest of the proof follows the same path as in (KcE).

(SE). The proof follows as for KcE.

(SA). Using EmpS twice, $\vdash (\mathsf{A}(\psi \rightarrow \chi) \wedge \mathsf{S}(\chi,\rho) \wedge \mathsf{A}(\rho \rightarrow \varphi)) \rightarrow (\mathsf{S}(\psi,\chi) \wedge \mathsf{S}(\chi,\rho) \wedge \mathsf{S}(\rho,\varphi))$. Then, using CompS twice we get $\vdash (\mathsf{S}(\psi,\chi) \wedge \mathsf{S}(\chi,\rho) \wedge \mathsf{S}(\rho,\varphi)) \rightarrow \mathsf{S}(\psi,\varphi)$. Therefore, by transitivity of the implication, $\vdash (\mathsf{A}(\psi \rightarrow \chi) \wedge \mathsf{S}(\chi,\rho) \wedge \mathsf{A}(\rho \rightarrow \varphi)) \rightarrow \mathsf{S}(\psi,\varphi)$ follows. □

Proof. (Prop. 4.4) We will show the case for $\mathsf{X} = \mathsf{Kc}_i$. Let $\Delta_1, \Delta_2 \in \mathsf{S}^\Gamma$, we prove $(\Delta_1)|_{\mathsf{Kc}} = (\Delta_2)|_{\mathsf{Kc}}$ by double inclusion.

($\subseteq$) Suppose $\mathsf{Kc}_i(\psi,\varphi) \in (\Delta_1)|_{\mathsf{Kc}}$. Since Δ_1 is a MCS, by 4KcA we have $\mathsf{AKc}_i(\psi,\varphi) \in (\Delta_1)|_{\mathsf{Kc}}$. Thus, by Def. 4.2, we get that $\mathsf{AKc}_i(\psi,\varphi) \in \Gamma|_{\mathsf{Kc}}$, and then $\mathsf{AKc}_i(\psi,\varphi) \in (\Delta_2)|_{\mathsf{Kc}}$. By TA , we get that $\mathsf{Kc}_i(\psi,\varphi) \in (\Delta_2)|_{\mathsf{Kc}}$.

($\supseteq$) For this inclusion, we show the counterpositive. Suppose $\mathsf{Kc}_i(\psi,\varphi) \notin (\Delta_1)|_{\mathsf{Kc}}$. Since Δ_1 is a MCS, $\neg\mathsf{Kc}_i(\psi,\varphi) \in (\Delta_1)|_{\mathsf{Kc}}$. By 5KcA, we have $\mathsf{A}\neg\mathsf{Kc}_i(\psi,\varphi) \in (\Delta_1)|_{\mathsf{Kc}}$, and by Def. 4.2, $\neg\mathsf{Kc}_i(\psi,\varphi) \in \Gamma|_{\mathsf{Kc}}$. Then $\mathsf{A}\neg\mathsf{Kc}_i(\psi,\varphi) \in (\Delta_2)|_{\mathsf{Kc}}$, and by TA, $\neg\mathsf{Kc}_i(\psi,\varphi) \in (\Delta_2)|_{\mathsf{Kc}}$. Therefore, $\mathsf{Kc}_i(\psi,\varphi) \notin (\Delta_2)|_{\mathsf{Kc}}$.

The case of $\mathsf{X} = \mathsf{S}$ (resp., $\mathsf{X} = \mathsf{O}$) are similar by using 4SA and 5SA (resp., 4NA and 5NA). □

Proof. (Prop. 4.5) If Δ has an $\mathsf{R}^\Gamma_{\langle\psi,\varphi\rangle}$-successor, then $\psi \in \Delta$ and $\mathsf{S}(\psi,\varphi) \in \Gamma$. Hence, every $\Delta' \in \mathsf{S}^\Gamma$ with $\varphi \in \Delta'$ is s.t. $(\Delta,\Delta') \in \mathsf{R}^\Gamma_{\langle\psi,\varphi\rangle}$. □

Proof. (Prop. 4.6) We start by extending Def. 4.2 with the following:

$$\Gamma|_{\neg\mathsf{X}} = \{\neg\mathsf{X}(\psi,\varphi) \mid \neg\mathsf{X}(\psi,\varphi) \in \Gamma\}$$
$$\Gamma|_{\neg\mathsf{Kc}} = \bigcup\{\Gamma|_{\neg\mathsf{Kc}_i} \mid i \in \mathsf{Agt}\}$$
$$\Gamma|_{\neg\mathsf{A}} = \{\neg\mathsf{A}\varphi \mid \mathsf{A}\varphi \in \Gamma\}$$

Now, some facts for any Δ in $\mathsf{S}^\Gamma \subseteq \mathbf{\Phi}$. By definition, $\Delta|_{\mathsf{S}} \cup \Delta|_{\neg\mathsf{S}}$ is a subset of Δ, and therefore it is consistent. Moreover: any of its maximally consistent extensions, say Δ', should satisfy $\Delta|_{\mathsf{S}} = \Delta'|_{\mathsf{S}}$. For ($\subseteq$), note that $\mathsf{S}(\psi,\varphi) \in \Delta|_{\mathsf{S}}$ implies $\mathsf{S}(\psi,\varphi) \in (\Delta|_{\mathsf{S}} \cup \Delta|_{\neg\mathsf{S}})$, and thus $\mathsf{S}(\psi,\varphi) \in \Delta'$, i.e., $\mathsf{S}(\psi,\varphi) \in \Delta'|_{\mathsf{S}}$. For ($\supseteq$), use the contrapositive. If $\mathsf{S}(\psi,\varphi) \notin \Delta|_{\mathsf{S}}$ then $\mathsf{S}(\psi,\varphi) \notin \Delta$, so $\neg\,\mathsf{S}(\psi,\varphi) \in \Delta$ (as Δ is a MCS). Thus, $\neg\,\mathsf{S}(\psi,\varphi) \in (\Delta|_{\mathsf{S}} \cup \Delta|_{\neg\mathsf{S}})$ and hence $\neg\,\mathsf{S}(\psi,\varphi) \in \Delta'$; therefore, $\mathsf{S}(\psi,\varphi) \notin \Delta'$ and thus $\mathsf{S}(\psi,\varphi) \notin \Delta'|_{\mathsf{S}}$.

For the proof of the proposition, suppose $\varphi \in \Delta$ for every $\Delta \in \mathsf{S}^\Gamma$. Take any $\Delta \in \mathsf{S}^\Gamma$, and note how $\Delta|_{\mathsf{S}} = \Gamma|_{\mathsf{S}}$. Then, the set $\Delta|_{\mathsf{S}} \cup \Delta|_{\neg\mathsf{S}} \cup \{\neg\varphi\}$ is inconsistent. Otherwise it could be extended into a MCS $\Delta' \in \mathbf{\Phi}$. By the result in the previous paragraph, this would imply $\Delta'|_{\mathsf{S}} = \Delta|_{\mathsf{S}}$, so $\Delta'|_{\mathsf{S}} = \Gamma|_{\mathsf{S}}$ and therefore $\Delta' \in \mathsf{S}^\Gamma$. But then, by the assumption, $\varphi \in \Delta'$, and by construction, $\neg\varphi \in \Delta'$. This would make Δ' inconsistent, a contradiction. Hence, there should be sets $\{\mathsf{S}(\psi_1,\varphi_1),\ldots,\mathsf{S}(\psi_n,\varphi_n)\} \subseteq \Delta|_{\mathsf{S}}$ and $\{\neg\,\mathsf{S}(\psi'_1,\varphi'_1),\ldots,\neg\,\mathsf{S}(\psi'_m,\varphi'_m)\} \subseteq \Delta|_{\neg\mathsf{S}}$ such that

$$\vdash \left(\bigwedge_{k=1}^{n} \mathsf{S}(\psi_k,\varphi_k) \wedge \bigwedge_{k=1}^{m} \neg\,\mathsf{S}(\psi'_k,\varphi'_k)\right) \rightarrow \varphi.$$

Hence, by Nec,

$$\vdash \mathsf{A}\left(\left(\bigwedge_{k=1}^{n} \mathsf{S}(\psi_k,\varphi_k) \wedge \bigwedge_{k=1}^{m} \neg\,\mathsf{S}(\psi'_k,\varphi'_k)\right) \rightarrow \varphi\right)$$

and then, by DistA and MP,

$$\vdash \mathsf{A}\left(\bigwedge_{k=1}^{n} \mathsf{S}(\psi_k,\varphi_k) \wedge \bigwedge_{k=1}^{m} \neg\,\mathsf{S}(\psi'_k,\varphi'_k)\right) \rightarrow \mathsf{A}\varphi.$$

Now, $\mathsf{S}(\psi_k,\varphi_k) \in \Delta|_{\mathsf{S}}$ implies, by 4SA and MP, that $\mathsf{A}\,\mathsf{S}(\psi_k,\varphi_k) \in \Delta$ for each $k = 1,\ldots,n$. Similarly, $\neg\,\mathsf{S}(\psi'_k,\varphi'_k) \in \Delta|_{\mathsf{S}}$ implies, by 5SA and MP, that $\mathsf{A}\neg\,\mathsf{S}(\psi'_k,\varphi'_k) \in \Delta$ for each $k = 1,\ldots,m$. Thus,

$$\bigwedge_{k=1}^{n} \mathsf{A}\,\mathsf{S}(\psi_k,\varphi_k) \in \Delta \qquad \text{and} \qquad \bigwedge_{k=1}^{m} \mathsf{A}\neg\,\mathsf{S}(\psi'_k,\varphi'_k) \in \Delta$$

and hence

$$\bigwedge_{k=1}^{n} \mathsf{A}\,\mathsf{S}(\psi_k,\varphi_k) \wedge \bigwedge_{k=1}^{m} \mathsf{A}\neg\,\mathsf{S}(\psi'_k,\varphi'_k) \in \Delta,$$

so

$$\mathsf{A}\left(\bigwedge_{k=1}^{n} \mathsf{S}(\psi_k,\varphi_k) \wedge \bigwedge_{k=1}^{m} \neg\,\mathsf{S}(\psi'_k,\varphi'_k)\right) \in \Delta$$

and therefore $\mathsf{A}\varphi \in \Delta$. □

Proof. (Prop. 4.7) Take any $\Delta \in \mathsf{S}^\Gamma$. On the one hand, if $\psi \in \Delta$ then, by the assumption, $(\Delta,\Delta') \in \mathsf{R}^\Gamma_{\langle\psi',\varphi'\rangle}$ for some Δ'. Hence, from $\mathsf{R}^\Gamma_{\langle\psi',\varphi'\rangle}$'s definition, $\psi' \in \Delta$ and thus (maximal consistency) $\psi \rightarrow \psi' \in \Delta$. On the other hand, if $\psi \notin \Delta$ then $\neg\psi \in \Delta$ (again, maximal consistency) and thus $\psi \rightarrow \psi' \in \Delta$. Thus, $\psi \rightarrow \psi' \in \Delta$ for every $\Delta \in \mathsf{S}^\Gamma$; then, by Prop. 4.6, $\mathsf{A}(\psi \rightarrow \psi') \in \Delta$ for every $\Delta \in \mathsf{S}^\Gamma$. □

Proof. (Prop. 4.8) Suppose that there is no $\Theta' \in \mathsf{S}^\Gamma$ such that $\varphi \in \Theta'$. Then, by maximal consistency, for all $\Theta' \in \mathsf{S}^\Gamma$, $\neg\varphi \in \Theta'$. By Prop. 4.6, for all $\Theta' \in \mathsf{S}^\Gamma$, $\mathsf{A}\neg\varphi \in \Theta'$ and by EmpS, $\mathsf{S}(\varphi,\bot) \in \Theta'$. Using KcN and NS when needed, $\mathsf{S}(\psi,\varphi) \in \Theta'$. By CompS, $\mathsf{S}(\psi,\bot) = \mathsf{A}\neg\psi \in \Theta'$ for all $\Theta' \in \mathsf{S}^\Gamma$ which using TA and MP means that $\neg\psi \in \Theta'$ for all $\Theta' \in \mathsf{S}^\Gamma$. Thus, $\psi \notin \Theta$, but that is a contradiction. Hence, there is $\Theta' \in \mathsf{S}^\Gamma$ such that $\varphi \in \Theta'$. □

Proof. (Lemma 4.1, Truth Lemma) Below we complete the proof on the Truth Lemma, showing the cases of the S and N operators. **Case $\varphi = \mathsf{S}(\psi,\rho)$:**. ($\Rightarrow$) Suppose that $\mathfrak{M}^\Gamma_c, \Theta \Vdash \mathsf{S}(\psi,\rho)$, thus (by $\Vdash$) there exists $\pi \in (\mathsf{Act}^\Gamma)^*$ s.t. $[\![\psi]\!]^{\mathfrak{M}^\Gamma_c} \subseteq \mathsf{SE}(\pi)$ and $\mathsf{R}_\pi([\![\psi]\!]^{\mathfrak{M}^\Gamma_c}) \subseteq [\![\rho]\!]^{\mathfrak{M}^\Gamma_c}$ We need to consider three cases:

- If $[\![\psi]\!]^{\mathfrak{M}_c^\Gamma} = \emptyset$, then by IH, $\neg\psi \in \Theta'$ for all $\Theta' \in S^\Gamma$. By Prop. 4.6, $A\neg\psi \in \Theta'$ for all $\Theta' \in S^\Gamma$ and therefore, $S(\psi, \bot) \in \Theta'$. Since $\bot \to \rho$ and $\rho \to \rho$ are tautologies, by Nec, $A(\bot \to \rho)$ and $A(\psi \to \psi)$ are in Θ'. Using an instance of SA in Prop. 4.1, $(A(\psi \to \psi) \wedge S(\psi, \bot) \wedge A(\bot \to \rho)) \to S(\psi, \rho) \in \Theta'$, by MP, $S(\psi, \rho) \in \Theta'$ for all $\Theta' \in S^\Gamma$. Thus, $S(\psi, \rho) \in \Theta$.
- If $[\![\psi]\!]^{\mathfrak{M}_c^\Gamma} \neq \emptyset$ and $\pi = \epsilon$, for all $\Theta' \in S^\Gamma$: if $\Theta' \in [\![\psi]\!]^{\mathfrak{M}_c^\Gamma}$ then $\Theta' \in [\![\rho]\!]^{\mathfrak{M}_c^\Gamma}$, i.e., for all $\Theta' \in S^\Gamma$, $\Theta' \notin [\![\psi]\!]^{\mathfrak{M}_c^\Gamma}$ or $\Theta' \in [\![\rho]\!]^{\mathfrak{M}_c^\Gamma}$. By IH, $\psi \notin \Theta'$ or $\rho \in \Theta'$. Since Θ' is maximally consistent, $\psi \to \rho \in \Theta'$ for all $\Theta' \in S^\Gamma$. By Prop. 4.6, $A(\psi \to \rho) \in \Theta'$ for all $\Theta' \in S^\Gamma$ and by EmpS, $S(\psi, \rho) \in \Theta'$. Thus, $S(\psi, \rho) \in \Theta$.
- If $[\![\psi]\!]^{\mathfrak{M}_c^\Gamma} \neq \emptyset$ and $\pi = \langle\psi_1, \varphi_1\rangle \ldots \langle\psi_n, \varphi_n\rangle$, we first prove by induction the following property (called $P(k)$), for any $k \leq n$: (1) $S(\psi, \varphi_k) \in \Gamma$ and (2) each φ_k-state is reached via $\langle\psi_1, \varphi_1\rangle \ldots \langle\psi_k, \varphi_k\rangle$ from some ψ-state.
  - $P(1)$: By the semantics of S, each ψ-state has an outgoing $\langle\psi_1, \varphi_1\rangle$-transition, and by Prop. 4.5, every φ_1-state can be reached from a ψ-state via $\langle\psi_1, \varphi_1\rangle$ (proving (2)). Also, by Prop. 4.7, we get that $A(\psi \to \psi_1) \in \Theta'$, for all $\Theta' \in S^\Gamma$. Thus, by EmpS, $S(\psi, \psi_1) \in \Theta'$. Since we established that $R^\Gamma_{\langle\psi_1,\varphi_1\rangle} \neq \emptyset$, using Prop. 4.2 it must be the case that $S(\psi_1, \varphi_1) \in \Gamma$ and therefore $S(\psi_1, \varphi_1) \in \Theta'$. Then by CompS and Prop. 4.4, $S(\psi, \varphi_1) \in \Theta'|_S = \Gamma|_S \subseteq \Gamma$, which proves (1).
  - $P(i-1) \to P(i)$: By IH, $S(\psi, \varphi_{i-1}) \in \Gamma$ and all φ_{i-1}-states are reached from some ψ-state via $\langle\psi_1, \varphi_1\rangle \ldots \langle\psi_{i-1}, \varphi_{i-1}\rangle$. Since π is SE at all ψ-states, each φ_{i-1}-state has some $R^\Gamma_{\langle\psi_i,\varphi_i\rangle}$-successor. By Prop. 4.7, $A(\varphi_{i-1} \to \psi_i) \in \Gamma$ and by EmpS, $S(\varphi_{i-1}, \psi_i) \in \Gamma$. By definition of R^Γ, $S(\psi_i, \varphi_i) \in \Gamma$. Thus, we have $S(\psi, \varphi_{i-1}), S(\varphi_{i-1}, \psi_i), S(\psi_i, \varphi_i) \in \Gamma$. Using CompS twice and MP, $S(\psi, \varphi_i) \in \Gamma$, proving (1). Since each φ_{i-1}-state has some $R^\Gamma_{\langle\psi_i,\varphi_i\rangle}$-successor, by Prop. 4.5, all φ_i-states can be $R^\Gamma_{\langle\psi_i,\varphi_i\rangle}$-reached from φ_{i-1}-states. Thus, using the IH, all φ_i-states can be reached via $\langle\psi_1, \varphi_1\rangle \ldots \langle\psi_i, \varphi_i\rangle$ from some ψ-state, which completes the proof of (2).

  We continue now with the proof of the main lemma. Let $k = n$, property $P(n)$ tells us that (1) $S(\psi, \varphi_n) \in \Gamma$, and (2) each φ_n-state is reached via π from some ψ-state. Recall that $\pi = \langle\psi_1, \varphi_1\rangle \ldots \langle\psi_n, \varphi_n\rangle$ is a witness of the satisfiability of $S(\psi, \rho)$, then, for every φ_n-state $\Theta' \in S^\Gamma$, $\mathfrak{M}_c^\Gamma, \Theta' \Vdash \rho$. Thus, if $\varphi_n \in \Theta'$ then $\rho \in \Theta'$ (by IH). Then, for every $\Theta' \in S^\Gamma$, $\varphi_n \to \rho \in \Theta'$. Using Prop. 4.6, $A(\varphi_n \to \rho) \in \Theta'$ and by EmpS, $S(\varphi_n, \rho) \in \Theta'|_S \subseteq \Gamma$. Thus, by CompS, $S(\psi, \rho) \in \Gamma$ and $S(\psi, \rho) \in \Theta$.

($\Leftarrow$) Suppose $S(\psi, \rho) \in \Theta$. Then, by Prop. 4.4, $S(\psi, \rho) \in \Theta'$ for all $\Theta' \in S^\Gamma$. Moreover, $S(\psi, \rho) \in \Gamma$ and $R^\Gamma_{\langle\psi,\rho\rangle}$ is defined. To prove that $\mathfrak{M}_c^\Gamma, \Theta \Vdash S(\psi, \rho)$ we have to consider two possibilities:

- There is no Θ' such that $\psi \in \Theta'$: By IH, $[\![\psi]\!]^{\mathfrak{M}_c^\Gamma} = \emptyset$. Using $\pi = \epsilon$, we trivially have that $\mathfrak{M}_c^\Gamma, \Theta \Vdash S(\psi, \rho)$.
- There is Θ' such that $\psi \in \Theta'$: by Prop. 4.8, there is Θ'' such that $\rho \in \Theta''$. By IH, $\mathfrak{M}_c^\Gamma, \Theta' \Vdash \psi$ and $\mathfrak{M}_c^\Gamma, \Theta'' \Vdash \rho$. Since it is defined, $\pi = \langle\psi, \rho\rangle$ is strongly executable at all ψ-states (since there is an $R^\Gamma_{\langle\psi,\rho\rangle}$-successor Θ'') and reaches from these only ρ-states via π (by construction of $R^\Gamma_{\langle\psi,\rho\rangle}$). Thus, $\mathfrak{M}_c^\Gamma, \Theta \Vdash S(\psi, \rho)$.

Case $\varphi = N(\psi, \rho)$**:**. ($\Rightarrow$) Suppose $\mathfrak{M}_c^\Gamma, \Theta \Vdash N(\psi, \rho)$. Then, there exists a $\pi = \langle\psi', \varphi'\rangle \in N^\Gamma$ such that $[\![\psi]\!]^{\mathfrak{M}_c^\Gamma} \subseteq SE(\pi)$ and $R_\pi([\![\psi]\!]^{\mathfrak{M}_c^\Gamma}) \subseteq [\![\rho]\!]^{\mathfrak{M}_c^\Gamma}$.

- If $[\![\psi]\!]^{\mathfrak{M}_c^\Gamma} = \emptyset$, then by IH, $\neg\psi \in \Theta'$ for all $\Theta' \in S^\Gamma$. by Prop. 4.6, $A\neg\psi \in \Theta'$ for all $\Theta' \in S^\Gamma$ and therefore $A(\psi \to \bot) \in \Theta'$. Since $\bot \to \rho$ is a tautology, by Nec, $A(\bot \to \rho) \in \Theta'$. Moreover, by Kc⊥ and using KcN, $N(\bot, \bot) \in \Theta'$. Using an instance of NA, $(A(\psi \to \bot) \wedge N(\bot, \bot) \wedge A(\bot \to \rho)) \to Kc_i(\psi, \rho) \in \Theta'$. By MP, we get $N(\psi, \rho) \in \Theta'$ for all $\Theta' \in S^\Gamma$. Thus, $N(\psi, \rho) \in \Theta$.
- If $[\![\psi]\!]^{\mathfrak{M}_c^\Gamma} \neq \emptyset$, by IH, for all $\Theta' \in S^\Gamma$, if $\psi \in \Theta'$ then:
  - Θ' has an $R^\Gamma_{\langle\psi',\varphi'\rangle}$-successor, and
  - for all $\Theta'' \in S^\Gamma$ s.t. $(\Theta', \Theta'') \in R^\Gamma_{\langle\psi',\varphi'\rangle}$, $\rho \in \Theta''$ (IH).

  By Prop. 4.7, $A(\psi \to \psi') \in \Theta'$ for all $\Theta' \in S^\Gamma$. By Prop. 4.5, every $\Theta'' \in S^\Gamma$ such that $\varphi' \in \Theta''$ can be $R^\Gamma_{\langle\psi',\varphi'\rangle}$-reached from Θ'. Therefore, for every $\Theta' \in S^\Gamma$ such that $\psi \in \Theta'$ we have that every $\Theta'' \in S^\Gamma$ such that $\varphi' \in \Theta''$ can be $R^\Gamma_{\langle\psi',\varphi'\rangle}$-reached from Θ' and by assumption, $\rho \in \Theta''$. Then, for every $\Theta'' \in S^\Gamma$ such that $\varphi' \in \Theta''$, we have $\rho \in \Theta''$. Thus, for all $\Theta'' \in S^\Gamma$, $\varphi' \to \rho \in \Theta''$. Using Prop. 4.6, for all $\Theta'' \in S^\Gamma$, $A(\varphi' \to \rho) \in \Theta''$. Finally, putting all together, for all $\Theta' \in S^\Gamma$, $\{A(\psi \to \psi'), N(\psi', \varphi'), A(\varphi' \to \rho)\} \subset \Theta'$. By axiom NA, $N(\psi, \rho) \in \Theta'$ and thus, $N(\psi, \rho) \in \Theta$.

($\Leftarrow$) Suppose $N(\psi, \rho) \in \Theta$, then by Prop. 4.4, $N(\psi, \rho) \in \Theta'$ for all $\Theta' \in S^\Gamma$. Moreover, $N(\psi, \rho) \in \Gamma$ and $R^\Gamma_{\langle\psi,\rho\rangle}$ is defined. As it happened with S and Kc_i, to prove that $\mathfrak{M}_c^\Gamma, \Theta \Vdash N(\psi, \rho)$, we have to consider two possibilities:

- There is no Θ' such that $\psi \in \Theta'$. By IH, $[\![\psi]\!]^{\mathfrak{M}_c^\Gamma} = \emptyset$. Since $N^\Gamma \neq \emptyset$, we choose any $\pi \in N^\Gamma$ as a witness and we trivially have that $\mathfrak{M}_c^\Gamma, \Theta \Vdash Kc_i(\psi, \rho)$.
- There is Θ' such that $\psi \in \Theta'$: by Prop. 4.8, there is Θ'' s.t. $\rho \in \Theta''$. By IH, $\mathfrak{M}_c^\Gamma, \Theta' \Vdash \psi$ and $\mathfrak{M}_c^\Gamma, \Theta'' \Vdash \rho$. Since it is defined, $\pi = \langle\psi, \rho\rangle$ is strongly executable at all ψ-states (since there is an $R^\Gamma_{\langle\psi,\rho\rangle}$-successor Θ'') and reaches from these only ρ-states via π (by construction of $R^\Gamma_{\langle\psi,\rho\rangle}$). Thus, $\mathfrak{M}_c^\Gamma, \Theta \Vdash N(\psi, \rho)$.

□

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In this section we prove that the SAT problem for DLKc is PSpace-complete. First, we show hardness.

Theorem 4. *DLKc is* PSpace-*hard.*

Proof. First we introduce some necessary definitions. Given a propositional formula ϕ with propositional variables $x_0, \dots, x_n$, a truth assignment for it can be written as a string $w \in \{\bot, \top\}^n$. Furthermore, $\phi[x \leftarrow b]$ denotes the formula obtained from ϕ when the propositional letter x is replaced by b, with $b \in \{\bot, \top\}$. We also introduce the needed notation for strings, we just use juxtaposition for concatenation. Furthermore, for any string w, $|w|$ denotes its length, and for $i < |w|$, $w[i]$ denotes the symbol at position i (starting at 0). We say that $w \Vdash \phi$ iff $w[x_0 \leftarrow w[0], \dots, x_n \leftarrow w[n]] = \top$. A Boolean tree of depth n is a set $t \subseteq 2^{\{\top,\bot\}^*}$ such that, $\epsilon \in t$; and if $wb \in t$, for some $w \in \{\bot, \top\}^*$ and $b \in \{\bot, \top\}$, then $w' \in t$ (that is, it is prefix closed).

Let us note that, given a QBF $Q_1 x_1 \dots Q_n x_n : \psi$, we have $Q_1 x_1 \dots Q_n x_n : \psi = \top$ iff there is a Boolean tree t such that, if $Q_i = \exists$ and $w \in t$ with $|w| = i$, then either $w\bot \in t$ or $w\top \in t$. If $Q_i = \forall$ then, if $w \in t$ with $|w| = i$, then both $w\bot \in t$ and $w\top \in t$. Furthermore, for every maximal $w \in t$ we have $w \Vdash \psi$.

We translate any QBF ϕ formula to a DLKc formula ψ, in such a way that $\phi = \top$ iff ψ is SAT. Consider a QBF $Q_1 x_1 : Q_2 x_2 : \dots Q_n X_n : \phi(x_0, \dots, x_n)$, where $Q_i \in \{\forall, \exists\}$, we consider fresh variables $y_0, \dots, y_n$ and and $z_0, \dots, z_{\lfloor \log n \rfloor}$. Using these variables, we consider a formula that captures the Boolean representation of a number. Let $bin(n)$ denote the binary presentation of number n, then we define formula:

$$B_i = \bigwedge_{0 \le k \le \lfloor \log n \rfloor \wedge bin(i)[k]=1} z_k \wedge \bigwedge_{0 \le k \le \lfloor \log n \rfloor \wedge bin(i)[k]=0} \neg z_k$$

We define a formula ψ as the conjunction of the following formulas:

ψ_0. $\mathsf{E}(y_0)$,
ψ_1. $\mathsf{A}(y_i \equiv B_i)$, for all $1 \le i \le n$,
ψ_2. $\mathsf{S}(y_i \wedge x_i, y_{i+1} \wedge x_i) \wedge \mathsf{S}(y_i \wedge \neg x_i, y_{i+1} \wedge \neg x_i)$, for all $1 \le i \le n$,
ψ_3. $(\mathsf{E}(y_i \wedge x_i,) \to \neg\,\mathsf{S}(y_i \wedge x_i, \neg x_i)) \wedge (\mathsf{E}(y_i \wedge x_i,) \to \neg\,\mathsf{S}(y_i \wedge \neg x_i, x_i))$, for all $1 \le i \le n$,
ψ_4. $\mathsf{S}(y_i, y_{i+1} \wedge \neg x_{i+1}) \wedge \mathsf{S}(y_i, y_{i+1} \wedge x_{i+1})$, for all $1 \le i \le n$ s.t. $Q_i = \forall$,
ψ_5. $\mathsf{A}(y_n \to \phi(x_0, \dots, x_n))$.

It is direct to see that this formula can be written in LSpace; for generating the formula one only needs to keep an index to count the number of quantifiers. Now, we prove that $Q_1 x_1 : Q_2 x_2 : \dots Q_n X_n : \phi(x_0, \dots, x_n) = \top$ iff ψ is SAT.

Only If) Suppose that $Q_1 x_1 : Q_2 x_2 : \dots Q_n X_n : \phi(x_0, \dots, x_n) = \top$ holds. We construct a model for ψ as follows. $\mathfrak{M} = (W, R, V)$, where $W \subseteq \{0, 1\}^*$ is defined as follows:

- $\epsilon \in W$,
- if $w \in W$ and $|w| = i < n$ then:
  - If $Q_i = \forall$, then $\{w0, w1\} \subseteq W$,
  - If $Q_i = \exists$, and $Q_{i+1} x_{i+1} \dots Q_n x_n : \Phi[X_0 \leftarrow w[0], \dots, X_{i-1} \leftarrow w[i-1], X_i \leftarrow \bot] = \top$, then $\{w0\} \subseteq W$,
  - If $Q_i = \exists$, and $Q_{i+1} x_{i+1} \dots Q_n x_n : \Phi[X_0 \leftarrow w[0], \dots, X_{i-1} \leftarrow w[i-1], X_i \leftarrow \top] = \top$, then $\{w1\} \subseteq W$,
- No other string belongs to W.

Now, V is defined as follows, fir any $w \in W$:

- $y_i \in V(w)$ iff $|w| = i$,
- $z_k \in V(w)$ iff $bin(|w|)[k] = 1$,
- If $|w| \ge i$ and $w.i = 0$ then $x_i \notin V(w)$,
- If $|w| \ge i$ and $w.i = 1$, then $x_i \in V(w)$,
- If $|w| < i$ then $x_i \notin V(w)$.

On the other hand, $A = \{0, 1\}$, and R is defined as follows. Let $w, w' \in W$, $w\ R_0\ w'$ iff $w' = w0$, and $w\ R_1\ w'$ iff $w' = w1$. First note that the model is deterministic, thus for any action $\pi \in \{0, 1\}$ and state w, it either denotes a unique path from w, or it is not strongly executable.

Pablo: Por favor chequear que esto este bien

Now, we prove that: $(R, W, V) \models \psi$, this boils down to prove that the model makes true all ψ_i.

- For formula $\mathsf{E}(y_0)$, we have that $y_0 \in V(\epsilon)$, by definition of V, and thus $(R, W, V) \models \mathsf{E}(y_0)$.
- For $\mathsf{A}(y_i \equiv \sigma_{i,0} z_0 \wedge \sigma_{i,1} z_1 \wedge \dots \wedge \sigma_{i,\lfloor \log n \rfloor} z_{\lfloor \log n \rfloor})$, where $\sigma_{i,0} z_0 \wedge \sigma_{i,1} z_1 \wedge \dots \wedge \sigma_{i,\lfloor \log n \rfloor} z_{\lfloor \log n \rfloor} = B_i$, and $\sigma_{i,k}$ (for $0 \le k \le \lfloor \log n \rfloor$) is $\neg$ or blank. If $\mathfrak{M}, w \Vdash y_i$, then $|w| = i$. If (for any k) $\sigma_{i,k} = \neg$ then $bin(|w|)[k] = 0$, hence by the definition above we have $\mathfrak{M}, w \Vdash \neg z_k$, similarly if $\sigma_{i,k}$ is blank we have $\mathfrak{M}, w \Vdash z_k$, thus we have $\mathfrak{M}, w \Vdash B_i$. The other direction is similar.
- For $\mathsf{S}(y_i, y_{i+1})$ (for any $1 \le i < n$), we have that $y_i \in V(w)$ for $|w| = i$ and if $Q_i = \exists$, then we have $w' \in W$ such that $w' = wb$ (for $b \in \{0, 1\}$), and furthermore $R_b(w) = \{w'\}$ and $y_{i+1}\ inV(w')$, thus $(R, W, V) \models \mathsf{S}(y_i, y_i + 1)$.
- For $\neg\,\mathsf{S}(y_i \wedge x_i, \neg x_i) \wedge \neg\,\mathsf{S}(y_i \wedge \neg x_i, x_i)$ for any i. Consider the case $\neg\,\mathsf{S}(y_i \wedge x_i, \neg x_i)$, and any $\pi \in \{0, 1\}^*$. If $[\![y_i]\!]^{\mathfrak{M}} \cap W \setminus \mathrm{SE}(\pi) \neq \emptyset$, the property follows. Otherwise, we must prove that $R_\pi([\![y_i \wedge x_i]\!]^{\mathfrak{M}}) \cap [\![x_i]\!]^{\mathfrak{M}} \neq \emptyset$. But note that y_i is only true at w if $|w| = i$, and $x_i \notin V(w)$ only if the ith bit of w is 0, in that case any for any extension w' of w we have $x_i \notin V(w')$, proving that $R_\pi([\![y_i \wedge x_i]\!]^{\mathfrak{M}}) \cap [\![x_i]\!]^{\mathfrak{M}} \neq \emptyset$. For $\neg\,\mathsf{S}(y_i \wedge \neg x_i, x_i)$ the proof is similar.

- For $\mathsf{S}(y_i, y_{i+1} \wedge \neg x_i) \wedge \mathsf{S}(y_i, y_{i+1} \wedge x_i)$. Let us prove that $(W, R, V) \models \mathsf{S}(y_i, y_{i+1} \wedge \neg x_i)$, we have that $y_i \in V(w)$ iff $|w| = i < n$, and since $Q_i = \forall$ by definition of R we have $w\ R\ w0$, and also $R_0(w) = \{w0\}$ thus $R_0([\![y_i]\!]^{\mathfrak{M}}) \subseteq [\![\neg x_i]\!]^{\mathfrak{M}}$ and also $R_0([\![y_i]\!]^{\mathfrak{M}}) \subseteq [\![y_{i+1}]\!]^{\mathfrak{M}}$ since $|w0| = i + 1$, thus $R_0([\![y_i]\!]^{\mathfrak{M}}) \subseteq [\![\neg x_i \wedge y_{i+1}]\!]^{\mathfrak{M}}$. For $\mathsf{S}(y_i, y_{i+1} \wedge x_i)$ the proof is similar.
- For $\mathsf{A}(y_n \to \phi(x_0, \dots, x_n))$, if $y_n \in V(w)$ then $|w| = n$, thus we have that w encodes a valuation for the Boolean formula ψ, by the sake of contradiction suppose that $\psi[x_0 \leftarrow w[0] \dots x_m \leftarrow w[m]] = \bot$. But by definition $w \notin W$, which is a contradiction.

Now, we prove that if $(R, W, V) \models \psi$, then $Q_1 x_1 : Q_2 x_2 : \dots Q_n X_n : \phi(x_0, \dots, x_n) = \top$, for doing so we define a binary tree $t \subseteq 2^{\{0,1\}^*}$ encoding the valuation that makes the formula true. Assume that we have a model such that $(W, R, V) \models \psi$.

By ψ_0 we have some state $w_0 \in W$ such that $\psi_0 \in V(w_0)$. Now, we define a Boolean tree $t \subseteq 2^{\{0,1\}^*}$ and a relation $T \subseteq W \times t$ For doing so, we consider Alg. 3. Note that the loop of line 3 has the following invariants:

$$s \in t \wedge |s| < i \wedge Q_{|s|} = \exists \to s\top \in t \vee s\bot \in t \tag{1}$$

$$s \in t \wedge |s| < i \wedge Q_{|s|} = \forall \to s\top \in t \wedge s\bot \in t \tag{2}$$

$$\forall (w, s) \in T \to (\forall k \leq |s| : V(x_k) \in w \equiv s[k] = \top) \tag{3}$$

Let us prove $(inv1 : alg : cap)$, when $i = 1$ the proof is direct because it holds for $|s| = 0$. For some $i > 0$, we need to take into account that when a $|s| = j < i$ is added to t if $Q_{|s|} = \exists$ than by ψ_3 and psi_4 always an element satisfying line 6 can be found thus by line 7 the added string satisfies 1. The proof for 2 is similar.

For $i = m$, and $s \in t$ and $|s| = n$ we have that $s \Vdash \psi(x_1, \dots, x_n)$. That is, $Q_1 x_1 \dots Q_n x_n : \psi(x_1, \dots, x_n)$. ◻

Algorithm 1 Algorithm for Extracting a Boolean tree from a model (W, R, V)

Input: A model (R, W, V) with $(R, W, V) \models \psi$
Output: A tree $t \subseteq 2^{\{0,1\}^*}$ s.t. t is a witness of ϕ

1: $t \leftarrow \{\epsilon\}$
2: $T^0 \leftarrow \{(w_0, \epsilon)\}$
3: **for** $i = 1$ to n **do**
4: **for** all $(w, s) \in T^{i-1}$ **do**
5: **if** $Q_i = \exists$ **then**
6: Choose w' s.t.
$$\exists \pi \in \{0,1\}^* : R_\pi(w) \wedge y_i \in V(w') \wedge \forall k < i : V(x_k) \in w \equiv V(x_k) \in w'$$
7: $t \leftarrow t \cup \{s(x_i \in V(w')?\top{:}\bot)\}$
8: $T^i \leftarrow T \cup \{(w', s(x_i \in V(w')?\top{:}\bot))\}$
9: **end if**
10: **if** $Q_i = \forall$ **then**
11: Choose w'_0 s.t.
$$\exists \pi : R_\pi(w) \wedge y_i \in V(w') \wedge x_i \notin V(w'_0) \wedge \forall k < i : V(x_k) \in w \equiv V(x_k) \in w'_0$$
12: Choose w'_1 s.t.
$$\exists \pi : R_\pi(w) \wedge y_i \in V(w') \wedge x_i \in V(w'_1) \wedge \forall k < i : V(x_k) \in w \equiv V(x_k) \in w'_0$$
13: $t \leftarrow t \cup \{s0, s1\}$
14: $T^i \leftarrow T^{i-1} \cup \{(w'_0, s\bot)\} \cup \{(w'_1, s\top)\}$
15: **end if**
16: **end for**
17: **end for**
18: **return** t

For proving PSpace-completeness we follows the ideas used in [], adapting the notions and concepts to our logic, this is not direct since in [] the logic operators are path operators and only paths in graphs need to take into account, here we need to take into account possible sequences of actions.

First, some auxiliary concepts and results are needed. A formula-labeled graph is a tuple (S, R, v) where S is a non-empty set of states, $R \subseteq S \times S'$ is a binary relation, and $v : S \to 2^{Form}$. A path in a labeled graph is a finite sequence of states: $s_0 s_1 \dots s_n$ such that $s_i\ R\ s_{i+1}$ for $0 \leq i \leq n - 1$.

Let ϕ be a formula in NNF, a *quasimodel* is a tuple graph $Q = (S, R, v, \{T_i\}_{i \in I})$, where (S, R, v) is a formula-labeled graph, and $\{T_i\}_{i \in I}$ is an indexed collection of non-empty binary relations $T_i \subseteq S \times S$ such that:

Q1 For every $i \in I$, $s\ T_i\ s'$ implies $s\ R^*\ s_;$.

Q2 For every $i, j \in I$ if $s\ T_i\ s'$ and $s'\ T_j s''$, then there is a $k \in I$ such that $s\ T_k s''$.

Furthermore, each $s \in S$ holds:

F1 if $\phi \wedge \psi \in v(s)$, then $\phi \in v(s)$ and $\psi \in v(s)$,
F2 if $\phi \vee \psi \in v(s)$, then $\phi \in v(s)$ or $\psi \in v(s)$,
F3 if $\mathrm{S}(\phi, \psi) \in v(s)$, then there is an $i \in I$, such that for all $s, s' \in S$ if $s\ T_i\ s'$ and $\phi \in v(s)$, then $\psi \in v(s')$.
F4 if $\neg\,\mathrm{S}(\phi, \psi) \in v(s)$, then for all $i \in I$ there is a pair $s, s' \in S$ such that $s\ T_i\ s'$ and $\phi \in v(s)$ and $\psi \notin v(s')$.

Furthermore, we define the notion of embedding. Given quasimodels $Q = (S, R, \{T_i\}_{i\in I}, v)$ and $Q = (S', R', \{T'_i\}_{j\in J}, v')$ an embedding is a couple of functions $e : S \to S'$, $\iota : I \to J$ such that $s\ R\ s'$ implies $e(s)\ R'^*\ e(s')$ and, for all $i \in I$ $e(T_i) \subseteq T'_f(i)$. We say that a quasimodel $(Q, R, v, \{T_i\}_{i\in I})$ is minimal if there is no quasimodel $Q' = (S', R', v, \{T'_j\}_{j\in J})$ embedded into Q such that $|S'| \leq |S|$ and $|R'| \leq |R|$ with at least one of these inequalities being strict. In the following theorem $\#_{\mathrm{S}}\phi$ denotes the number of S modalities appearing in ϕ. First, we prove the following lemma.

We note that a structure Q is a quasimodel for ϕ iff there is a minimal quasimodel Q' for ϕ that is embedded into Q.

Theorem 5. *A NNF formula ϕ is UNSAT iff for every minimal quasimodel $Q = (S, R, v, \{T_i\}_{i\in I})$ of ϕ there is a path $s_0 \dots s_n$ such that $n \leq \#_{\mathrm{S}}\phi + 1$, $\phi \in v(s_0)$ and there is a $0 \leq k \leq n$ such that s_k contains an inconsistent labeling.*

Proof. "If": This is the more direct part. By contrapositive we prove that, if a formula ϕ is SAT, then there is a quasimodel Q having no path $s_0 \dots s_n$ such that $n \leq \#_{\mathrm{S}}\phi + 1$, $\phi \in v(s_0)$ and a s_k with an inconsistent labeling. Suppose that ϕ is SAT, that is, there is a a model $M = (S, R, V)$, with $M \vDash \phi$, then we define a quasimodel $Q = (S^Q, R^Q, v^Q, \{T_i\}_{i\in I}\})$ as follows. First, we define the index set: $I = \{\pi \mid \pi \in \mathrm{Act}^* \wedge \exists s \in S : \pi \text{ is strongly executable at } s\}$. Then, we define $S^Q = S$, $R^Q = \bigcup_{\pi\in I} R_\pi$; and for $\pi \in I$: $T_\pi = R_\pi$. While, v is defined using V. That is $p \in v(s)$ iff $p \in V(s)$ for every $p \in \Phi$ and $s \in S$. The other cases are defined recursively, $\phi'' \wedge \psi'' \in v(s)$ iff $\phi'' \in v(s)$ and $\psi'' \in v(s)$, $\phi'' \vee \psi'' \in v(s)$ iff $\phi'' \in v(s)$ or $\psi'' \in v(s)$. For modalities we define $\mathrm{S}(\phi'', \psi'') \in v(s)$ iff $M \vDash \mathrm{S}(\phi'', \psi'')$, and $\neg\,\mathrm{S}(\phi'', \psi'') \in v(s)$ iff not $M \vDash \mathrm{S}(\phi'', \psi'')$. We need to prove that this definition satisfies the requirements to be a quasimodel. Condition **Q1** holds straightforwardly by definition. Condition **Q2** follows from the fact that the composition of two SE actions is itself SE. The conditions over v are met because it is defined via the truth valuation of a model.

"Only If": First, we prove that given a quasimodel $Q = (S, R, v, \{T_i\}_{i\in I})$ for a formula ϕ without inconsistent labels we can define a model $M = (S', R', V)$ such that $M \vDash \phi$. M is defined as follows, we assume a one-to-one mapping $f : I \to \mathrm{Act}$, it always exists because both sets are enumerable. $S' = \bigcup_{i\in I} T_i(S) \cup \bigcup_{i\in I} T_i^{-1}(S)$, for each $a \in \mathrm{Act}$ we set $R_a = T_i$ if $f(i) = a$, and $R_a = \emptyset$ if there is no $i \in I$ such that $f(i) = a$. v is defined as follows, $p \in V(s)$ iff $p \in v(s)$. Let us prove that $M, s \vDash \phi$ for some state s, if ϕ is propositional the proof is direct. Otherwise, assume that ϕ contains some modality. If $\mathrm{S}(\phi', \psi')$ is a subformula of ϕ, then $\mathrm{S}(\phi', \psi') \in v(s)$ (for some s), and therefore there is an $i \in I$ such that for all s, s' if $s\ T_i\ s'$ and $\phi' \in v(s)$ then $\psi' \in v(s')$. This means that for $f(i) \in \mathrm{Act}^*$ we have $R_{f(i)}([\![\phi']\!]^M) \subseteq [\![\psi']\!]^M$, also it is SE ($f(i)$ consists only of an action symbol). If ϕ contains $\neg\,\mathrm{S}(\phi', \psi')$, then take any $\pi \in \mathrm{Act}^*$ if $R_\pi \neq \emptyset$ then it is defined by means of a composition of $T'_i s$, that is: $R_\pi = T_{i_0} \dots T_{i_n}$, that is, due to **Q2** there is a k such that: $R_\pi = T_{i_0} \dots T_{i_n} = T_k$ and therefore because of the definition of quasimodel we have that there is a pair $s, s' \in S$ such that $s\ T_k\ s'$ and $\phi \in v(s)$ and $\psi \notin v(s')$, thus $R_\pi([\![\phi']\!]^M) \cap [\![\neg\psi']\!]^M \neq \emptyset$. Summing up, $M \vDash \phi$.

To finish the proof, we prove the following claim:

Claim .1. *Let Q be any minimal quasimodel for a formula ϕ, then there is a $s_0 \in S$ sucht that $\phi \in v(s_0)$ and for any path $s_0 \dots s_n$ in Q we have that $|\{v(s) \mid \exists 0 \leq k \leq n : s = s_k\}| \leq \#_{\mathrm{S}}\phi + 1$ (that is, the number of disjoint labels in the path is linearly bounded by the number of modalities in ϕ).*

Having this claim at hand the result follows: if ϕ is UNSAT, then all the quasimodels of ϕ contains states with inconsistent labeling, otherwise we can construct a model (as above). But since, in minimal quasimodels, the number of different labels of any path is bounded by the length of the formula, we must have a path with an inconsistent label.

We prove the claim, we proceed by cases:

- If ϕ is a propositional formula the proof is direct.
- If $\phi = \mathrm{S}(\phi', \psi')$, let s be an state such that $\mathrm{S}(\phi', \psi') \in v(s)$, by definition of quasimodels, there must be an $i \in I$ such that $R_i([\![\phi']\!]^M) \subseteq [\![\psi']\!]^M$. We can define a new quasimodel $Q' = (R', \{(s, s')\}, V', \{R'\})$ where $R' = \{(s, s')\}$, $V'(s) = V(s)$ and $V'(s) = V(s)$. This is a quasimodel for $\mathrm{S}(\phi', \psi')$. Furthermore, it is embedded into Q. If Q has more states then it is not minimal (a contradiction), otherwise it only has two states and the result follows.
- If $\phi = \neg\,\mathrm{S}(\phi', \psi')$, the proof is as above, we select an arbitrary relation and show that a minimal quasimodel would have two states.
- If $\phi = \phi' \wedge \phi''$, by the sake of contradiction, suppose that for some $s_0 \in S$ we have $\phi \wedge \phi' \in v(s)$ and there is some path $s_0 \dots s_{n-1}$ such that $\#_{\mathrm{S}}\phi + 1 < |\{v(s_i) \mid 0 \leq i < n\}|$ that is $\#_{\mathrm{S}}\phi' + \#_{\mathrm{S}}\phi' + 1 < |\{s_i \mid 0 \leq i < n\}|$. Since $\{\phi, \phi'\} \subseteq v(s)$ there are quasimodels $Q_\phi, Q_{\phi'}$ of ϕ and ϕ', respectively, that are embedded into Q. By inductive hypothesis we now that all the paths in Q and Q' has at most $\max\{\#_{\mathrm{S}}\phi, \#_{\mathrm{S}}\phi'\} + 1$ disjoint labels, that is, there is some state that s_k such that does not embed any state of Q nor Q'. This state can be removed and the obtained structure would be a quasimodel of $\phi \wedge \phi'$, this is a contradiction since we assumed that Q is minimal.

□

Theorem 6. DLKc *is* PSpace*-complete.*

Proof. Using Theorem **??**, we give a PSpace algorithm that searches for paths in quasimodels of linear length that contain inconsistent labeling.

□